Chapter 7

Integrals

Integration by Substitution & Trignometric Identities

1. Integration by Substitution

Let g be a function whose range is an interval l, and let f be a function that is continuous on l. If g is differentiable on its domain and F is an antiderivative of f on l,

then $\int f(g(x))g'(x)\,dx = F(g(x)) + C$.

If $u = g(x)$, then $du = g'(x)$ and $\int f(u)\,du = F(u) + C$.

Guidelines for making a change of variable

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.

2. Compute $du = g'(x)\,dx$.

3. Rewrite the integral in terms of the variable u.

4. Evaluate the resulting integral in terms of u.

5. Replace u by g(x) to obtain an antiderivative in terms of x.

The General Poser Rule for Integration

If g is a differentiable function of x, then $\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1$

Rationalizing Substitutions

Some irrational functions can be changed into rational functions by means of appropriate substitutions.

In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Some Standard Substitutions

(i) $\int \frac{dx}{x(x^n+1)}$ $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

(ii) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$ $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(iii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$ take x^n common as x and put $1 + x^{-n} = t$.

(iv) $\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(v) $\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

2. Integration Using Trigonometric Identities

In the integration of a function, if the integrand involves any kind of trigonometric function, then we use trigonometric identities to simplify the function that can be easily integrated.

Few of the trigonometric identities are as follows:

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\cos^3 x = \frac{3\cos x + \cos 3x}{4}$$

All these identities simplify integrand, that can be easily found out.

Solved Examples:

Ex.1 Evaluate $\int (x^2+1)^2(2x)\,dx$.

Sol. Letting $g(x) = x^2 + 1$, we obtain $g'(x) = 2x$ and $f(g(x)) = [g(x)]^2$.

From this, we can recognize that the integrand and follows the f(g(x)) g'(x) pattern.

Thus, we can write $\int \overbrace{(x^2+1)^2}^{[g(x)]^2} \overbrace{(2x)}^{g'(x)} dx = \frac{1}{3}(x^2+1)^3 + C.$

Ex.2 Evaluate $\int \frac{-4x}{(1-2x^2)^2} dx$

Sol. $\int \overbrace{(1-2x^2)^{-2}}^{u^{-2}} \overbrace{(-4x)dx}^{du} = \frac{\overbrace{(1-2x^2)^{-1}}^{u^{-1}(-1)}}{-1} + C$

Ex.3 Evaluate $\int x^3 \cos(x^4+2)\, dx$.

Sol.

Let $u = x^4 + 2 \Rightarrow du = 4x^3\, dx$

$\int x^3 \cos(x^4+2)\, dx = \int \cos u . \frac{1}{4} du = \frac{1}{4}\int \cos u\, du$

$= \frac{1}{4}\sin u + C \quad = \frac{1}{4}\sin(x^4+2) + C$

Ex.4 Evaluate $\int \frac{x^2 dx}{(x^3-2)^5}$.

Sol.

Let $u = x^3 - 2$. Then $du = 3x^2\, dx$. so by substitution :

$\int \frac{x^2 dx}{(x^3-2)^5} = \int \frac{du/3}{u^5} = \frac{1}{5}\int u^{-5} du$

$= \frac{1}{3}\frac{u^{-4}}{-4} + C = -\frac{1}{12}(x^3-2)^{-4} + C.$

Ex.5 Evaluate $\int \frac{\sqrt{x+4}}{x} dx$

Sol. Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\, du$.

Therefore $\int \frac{\sqrt{x+4}}{x} dx = \int \frac{u}{u^2-4} 2u\, du$

$$= 2\int \frac{u^2}{u^2-4}du = 2\int \left(1+\frac{4}{u^2-4}\right)du$$

$$= 2\int du + 8\int \frac{du}{u^2-4} = 2u + 8 \cdot \frac{1}{2\cdot 2}\ \ell n\left|\frac{u-2}{u+2}\right| + C$$

$$= 2\sqrt{x+4} + 2\ \ell n\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$$

Ex.6 Evaluate $\int \frac{dx}{1+e^x}$

Sol. Rewrite the integrand as follows :

$$\frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}}\left(\frac{1}{1+e^x}\right) = \frac{e^{-x}}{e^{-x}+1} \quad (u = e^{-x} + 1;\ du = -e^{-x}\,dx)$$

$$\int \frac{dx}{1+e^x} \int \frac{e^{-x}dx}{e^{-x}+1} = \int \frac{-du}{u} = -\ell n|u| + C$$

$$= -\ln(e^{-x}+1) + c \quad (\because\ e^{-x}+1>0)$$

Ex.7 Evaluate $\int$ **sec x dx**

Sol. Multiply the integrand sec x by sec x + tan x and divide by the same quantity :

$$\int \sec x\,dx = \int \frac{\sec(\sec x+\tan x)}{\sec x+\tan x}dx = \int \frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}dx$$

Now put $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)\,dx$

we find $\int \frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}dx = \int \frac{du}{u} = \ell n\,|u| + C = \ell n|\sec x + \tan x| + C$

Ex.8 Evaluate $\int$ **cos x (4 - sin2 x) dx**

Sol. Put sin x = t so that cos x dx = dt. Then the given integral

$$= \int \sqrt{(4-t^2)}\,dt = \int \sqrt{(2^2-t^2)}\,dt$$

$$= \frac{1}{2}t\sqrt{(2^2-t^2)} + \frac{2^2}{2}\sin^{-1}(t/2) + c$$

$= \frac{1}{2} \sin x \cdot \sqrt{(4-\sin^2 x)} + 2 \sin^{-1} (1/2 \sin x) + c$

Ex.9 Integrate

(i) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$,

(ii) $\frac{10x^9 + 10^x . \log_e 10}{10^x + x^{10}}$

Sol.

(i) Let $I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$. Now putting $e^x + e^{-x} = t$, so that $(e^x - e^{-x})\, dx = dt$,

we have $I = \int (1/t)\, dt = \log t = \log (e^x + e^{-x})$.

(ii) Here $I = \int \frac{10x^9 + 10^x . \log_e 10}{10^x + x^{10}} dx$. Now putting $10^x + x^{10} = t$, and $(10^x \log_e 10 + 10x^9)\, dx = dt$,

we have $I = \int (1/t)\, dt = \log t = \log (10^x + x^{10}) + c$

Ex.10 Integrate

(i) $\frac{1}{x \cos^2 (1 + \log x)}$,

(ii) $\frac{1}{x(1 + \log x)^m}$

Sol.

(i) Here $I = \int dx/\{x\cos^2(1+\log x)\}$. Putting 1 + log x = t, so that (1/x) dx = dt, we have

$$I = \int dt/\cos^2 t = \int \sec^2 t\,dt = \tan t = \tan(1 + \log x) + c.$$

(ii) Here $I = \int dx/\{x(1+\log x)^m\}$. Putting 1 + log x = t, so that (1/x)dx = dt, we have

$$I = \int \frac{dt}{t^m} = \frac{t^{-m+1}}{-m+1} = \frac{(1+\log x)^{-m+1}}{(1-m)} = \frac{1}{(1-m)}(1 + \log x)^{1-m} + c.$$

Ex.11 Integrate

(i) $\frac{\cot x}{\log(\sin x)}$

(ii) $\frac{\tan x}{(\log(\sec x)}$

Sol.

(i) Here $\frac{d}{dx}(\log \sin x) = \frac{1}{\sin x}\cos x = \cot x.$

$$\therefore \quad I = \int \frac{\cot x\,dx}{\log \sin x} = \int \frac{\cot x}{t} \times \frac{dt}{\cot x} = \log|\log(\sin x)| + c$$

(ii) We have $\int \frac{\tan x\,dx}{\log \sec x} = \log|\log \sec x| + c$

Ex.12 Integrate $\sqrt{(1+\sin x)}$

Sol.

We have $I = \int \sqrt{(1+\sin x)}\,dx = \int \sqrt{\left\{1-\cos\left(\frac{\pi}{2}+x\right)\right\}}\,dx = \int \sqrt{\left[2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]}\,dx$

Now put $\frac{x}{2} + \frac{\pi}{4} = t \Rightarrow \frac{1}{2}dx = dt$ or $dx = 2\,dt$, we have $I = \int \sqrt{2\sin^2 t}\,(2dt) = -2\sqrt{2}\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)+c$

Ex.13 Integrate $\cos^5 x$.

Sol.

$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx$

$= \int (1-\sin^2 x)^2 \cos x\, dx = \int (1-t^2)^2\, dt$ [put $\sin x = t \Rightarrow \cos x\, dx = dt$]

$= \int (1-2t^2+t^4)\, dt = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + c$

$= \frac{\sin^5 x}{5} - \frac{2}{3}\sin^3 x + \sin x + c$

Ex.14 Evaluate $\int \frac{\cos^5 x}{\sin^2 x} dx$

Sol.

Let $I = \int \frac{\cos^5 x}{\sin^2 x} dx = \int \frac{\cos^4 x}{\sin^2 x} \cos x\, dx$

$= \int \frac{(1-\sin^2 x)^2}{\sin^2 x} \cos x\, dx$ [put $\sin x = t \Rightarrow \cos x\, dx = dt$]

then $I = \int \frac{(1-t^2)^2}{t^2} = dt \int \frac{1-2t^2+t^4}{t^2} dt$

$= \int \left[\frac{1}{t^2} - 2 + t^2\right] dt = -\frac{1}{t} - 2t + \frac{t^3}{3}$

$= -\frac{1}{\sin x} - 2\sin x + \frac{\sin^3 x}{3}$

$= -\operatorname{cosec} x - 2\sin x + \frac{1}{3}\sin^3 x + c$

Ex.15 Integrate $1/(\sin^3 x \cos^5 x)$.

Sol. Here the integrand is $\sin^{-3} x \cos^{-5} x$. It is of type $\sin^m x \cos^n x$, where $m + n = -3 - 5 = -8$ i.e., –ve even integer

$\therefore I = \int \frac{dx}{\sin^3 x \cos^5 x} = \int \frac{dx}{(\sin^3 x / \cos^3 x)\cos^3 x \cos^5 x}$

$= \int \frac{\sec^8 x dx}{\tan^3 x} = \int \frac{\sec^6 x . \sec^2 x dx}{(\tan^3 x)} = \int \frac{(1+\tan^2 x)^3 \sec^2 x dx}{\tan^3 x}$

Now put $\tan x = t$ so that $\sec^2 x\,dx = dt$

$$\therefore\quad I = \int \frac{(1+t^2)^3 dt}{t^3} = \int\left(\frac{1}{t^3}+\frac{3}{t}+3t+t^2\right) dt$$

$$= -\frac{1}{2t^2} + 3\log t + \frac{3}{2}\tan^2 x + \frac{1}{3}\tan^3 x$$

Ex.16 Integrate $1/\sqrt{(\cos^3 x \sin^5 x)}$

Sol. Here the integrand is of the type cosm x sinnx. We have m = –3/2, n = – 5/2, m + n = – 4 i.e., and even negative integer.

$$\therefore\quad \int \frac{dx}{\sqrt{(\cos^3 x \sin^5 x)}} = \int \frac{dx}{\cos^{3/2} x \sin^{5/2} x}$$

$$= \int \frac{dx}{\cos^{3/2} x(\sin^{5/2} x/\cos^{5/2} x)\cos^{5/2} x}$$

$$= \int \frac{dx}{\cos^4 x \tan^{5/2} x} = \int \frac{\sec^4 x}{\tan^{5/2} x} dx = \int \frac{\sec^2 x}{\tan^{5/2} x} \sec^2 x\, dx$$

$$= \int \frac{(1+\tan^2 x)}{\tan^{5/2} x} \sec^2 x\, dx = \int \frac{(1+t^2)}{t^{5/2}} dt,$$ putting $\tan x = t$ and $\sec^2 x\, dx = dt$

$$= \int (t^{-5/2} + t^{-1/2}) dt = -\frac{2}{3} t^{-3/2} + 2t^{1/2}$$

$$= -\frac{2}{3} (\tan x)^{-3/2} + 2(\tan x)^{1/2} = 2\sqrt{(\tan x)} - \frac{2}{3} (\tan x)^{-3/2} + c$$

Ex.17 Evaluate $\int \frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}}$

Sol.

Put $x - \beta = y \Rightarrow dx = dy$

Given integral

$$I = \int \frac{dy}{\sqrt{\cos^3 y \sin(y+\beta+\alpha)}}$$

$$\Rightarrow I = \int \frac{dy}{\sqrt{\cos^3 y \sin(y+\theta)}} \quad (\theta = \alpha + \beta)$$

$$= \int \frac{dy}{\sqrt{\cos^3 y(\sin y\cos\theta + \cos y \sin\theta)}}$$

$$= \int \frac{dy}{\sqrt{\cos^4 y(\cos\theta \tan y + \sin\theta)}}$$

$$= \int \frac{\sec^2 y dy}{\sqrt{(\cos\theta \tan y + \sin\theta)}}$$

Now put $\sin\theta + \cos\theta \tan y = z^2 \Rightarrow$ cosq $\sec^2 y\, dy = 2z\, dz$

$$\Rightarrow \quad I = \int \frac{2z\sec\theta dz}{z}$$

$$\Rightarrow \quad 2\sec\theta \sqrt{\frac{\sin(y+\theta)}{\cos y}} + c$$

$$= 2\sec(\alpha+\beta)\sqrt{\frac{\sin(x+\alpha)}{\cos(x-\beta)}} + c$$

Ex.18 Evaluate $\int \frac{5x^4 + 4x^5}{(x^5+x+1)^2}$ **dx**

Sol.

$$I = \int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx = \int \frac{x^4(5+4x)dx}{x^{10}\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}$$

$$= \int \frac{5/x^6 + 4/x^5}{\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2} dx,$$

put $1 + \frac{1}{x^4} + \frac{1}{x^5} = t$

$$\Rightarrow \left(-\frac{5}{x^5} - \frac{6}{x^6}\right) dx = dt = \int -\frac{dt}{t^2} = \frac{1}{t} + c$$

$$= \frac{1}{1+\frac{1}{x^4}+\frac{1}{x^5}} + c = \frac{x^5}{x^5+x+1} + c$$

Introduction to Integrals

Integral Calculus

- Integral Calculus is the branch of calculus where we study integrals and their properties.
- Integration is a very important concept which is the inverse process of differentiation. Both the integral calculus and the differential calculus are related to each other by the fundamental theorem of calculus.
- If we know the f' of a function that is differentiable in its domain, we can then calculate f. In differential calculus, we used to call f', the derivative of the function f. Here, in integral calculus, we call f as the anti-derivative or primitive of the function f'. And the process of finding the anti-derivatives is known as anti-differentiation or integration.
- Integration can be classified into two different categories:
  (i) Definite Integral
  (ii) Indefinite Integral

Definite Integral

- An integral that contains the upper and lower limits **i.e.** start and end value, then it is known as a definite integral. On a real line, x is restricted to lie. Definite Integral is also called a Riemann Integral when it is restricted to lie on the real line.
- A definite Integral is represented as: $\int_a^b f(x)dx$

Indefinite Integral

- Indefinite integrals are not defined using the upper and lower limits. It represents the family of the given function whose derivatives are f. It returns a function of the independent variable.
- The integration of a function f(x) is given by F(x) and it is represented by:
  $\int f(x)\,dx = F(x) + C$, where R.H.S. of the equation means integral of f(x) with respect to x.
  F(x) = Anti-derivative or primitive
  f(x) = Integrand
  dx = Integrating agent.
  C = Constant of integration.
  x = Variable of integration.
- It may seem strange that there exists an infinite number of anti-derivatives for a function f.
  Example: Let us take $f'(x) = 3x^2$.
  By hit and trial, we can find out that its anti-derivative is $F(x) = x^3$ because if you differentiate F with respect to x, you will get $3x^2$.
  There is only one function that we got as the anti-derivative of f(x).
  If we differentiate $G(x) = x^3 + 9$ with respect to x, we would get the same derivative **i.e.** f(x).
- This gives us an important insight. Since the differentiation of all the constants is zero, we can write any constant with F(x) and the derivative would still be equal to f(x). Thus, there are infinite constants that can be substituted for C in the equation
  Hence, there are infinite functions whose derivative is equal to f. C is called an arbitrary constant or the constant of integration.

Properties of Indefinite Integrals:

(i) The process of differentiation and integration are inverses of each other.

i.e. $\frac{d}{dx}\int f(x)\,dx = f(x)$ and $\int f'(x)\,dx = f(x) + C$

(ii) Two indefinite integrals with the same derivative lead to the same family of curves, and so they are equivalent.

(iii) The integral of the sum of two functions is equal to the sum of integrals of the given functions,

i.e. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(iv) For any real value of p, $\int pf(x)\,dx = p\int f(x)\,dx$

(v) For a finite number of functions $f_1, f_2 \ldots. f_n$ and the real numbers $p_1, p_2 \ldots p_n$,

$\int [p_1f_1(x) + p_2f_2(x) \ldots. + p_nf_n(x)]\,dx = p_1\int f_1(x)dx + p_2\int f_2(x)dx + \ldots.. + p_n\int f_n(x)dx$

Methods of Integration

1. Integration by substitution: In this method the integral $\int f(x)dx$ is expressed in terms of another integral where some other variables say t is the independent variable; x and t being connected by some suitable relation x=g(t).
It leads to the result $\int f(x)dx=\int f(g(t)).\, g'(t)\, dt$

2. Integration by parts: This method is used to integrate the product of two functions. If f(x) and g(x) are two integrable functions, then

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left[\frac{\mathrm{d}\,f(x)}{\mathrm{d}x}\int g(x)\,dx\right]dx$$

i.e. The integral of (product of two functions) = first function * integral of the second - integral of (derivative of first function * integral of the second function)

In order to select the first function, the following order is followed:

Inverse → Logarithmic → Algebraic → Trigonometric → Exponential

3. Integration by a partial fraction: If the integral is in the form of an algebraic fraction that cannot be integrated then the fraction needs to be decomposed into partial fractions.

Rules for expressing in partial fraction:

- The numerator must be at least one degree less than the denominator.
- For every factor (ax+b) in the denominator, there is a partial fraction $\frac{A}{ax+b}$
- If a factor is repeated in the denominator n times then that partial fraction should be written n times with degree 1 through n
- For a factor of the form (ax^2+bx+c) in the denominator, there will be a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$

Uses of Integral Calculus

Integral Calculus is mainly used for the following two purposes:

- To calculate f(x) from f'(x). If a function f is differentiable in the interval of consideration, then f' is defined in that interval. We have already seen in differential calculus how to calculate derivatives of a function. We can "undo" that with the help of integral calculus.

- To calculate the area under a curve.
  Until now, we have learned that areas are always positive. But as a matter of fact, there is something called a signed area.

Integral Calculus Formulas

We had differentiation formulas, we have integral formulas as well. Let us go ahead and look at some of the integral calculus formulas.

- $\int a\, dx = ax + C$
- $\int \frac{1}{x}\, dx = \ln|x| + C$
- $\int e^x\, dx = e^x + C$
- $\int a^x\, dx = \frac{e^x}{\ln a} + C$
- $\int \ln x\, dx = x \ln x - x + C$
- $\int \sin\ x\, dx = -\cos\ x + C$
- $\int \cos\ x\, dx = \sin\ x + C$
- $\int \tan\ dx = \ln|\sec\ x| + C\ or\ -\ln|\cos\ x| + C$
- $\int \cot\ x\, dx = \ln|\sin\ x| + C$
- $\int \sec\ x\, dx = \ln|\sec\ x + \tan\ x| + C$
- $\int \csc\ x\, dx = \ln|\csc\ x - \cot\ x| + C$
- $\int \sec^2\ x\, dx = \tan\ x + C$
- $\int \sec\ x\ \tan\ x\, dx = \sec\ x + C$
- $\int \csc^2\ x\, dx = -\cot\ x + C$
- $\int \tan^2\ x\, dx = \tan\ x - x + C$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
- $\int \frac{dx}{\sqrt{a^2 + x^2}} = \frac{1}{a}\arcsin\left(\frac{x}{a}\right) + C$

Application of Integral Calculus

The important application of integral calculus are as follow:

- The area between two curves
- Centre of mass
- Kinetic energy

- Surface area
- Work
- Distance, velocity and acceleration
- The average value of a function
- Volume
- Probability

Integral Calculus Examples

Example: Find the integral for the for the following function.
(i) f(x) = √x

$$F(x) = \int f(x)\,dx = \int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx$$

$$\because \int \sqrt{x}\,dx = \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\Rightarrow F(x) = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3}x^{\frac{3}{2}} + C$$

(ii) f(x) =cos²x

$$F(x) = \int cos^2 x\,dx = \frac{1}{2}\int 2cos^2 x\,dx = \frac{1}{2}\int (cos2x - 1)dx$$

$$\Rightarrow F(x) = \frac{1}{2}\left[\int cos2x\,dx - \int 1\,dx\right]$$

$$\because \int f(k) = \frac{F(x)}{k} + C \ and \ \int cos\ x\,dx = sin\ x + C$$

$$\therefore F(x) = \frac{1}{2}\left[\frac{sin2x}{2} + C_1 - (x + C_2)\right] = \frac{1}{2}\left[\frac{sin2x}{2} - x\right] + C$$

Integration by Partial Fractions

Introduction

We know that a Rational Number can be expressed in the form of p/q, where p and q are integers and q≠0. Similarly, a rational function is defined as the ratio of two polynomials which can be expressed in the form of partial fractions: P(x)/Q(x), where Q(x)≠0.

1. **Proper partial fraction:** When the degree of the numerator is less than the degree of the denominator, then the partial fraction is known as a proper partial fraction.
2. **Improper partial fraction:** When the degree of the numerator is greater than the degree of denominator then the partial fraction is known as an improper partial fraction. Thus, the fraction can be simplified into simpler partial fractions, that can be easily integrated.

In this section, we show how to integer any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called partial fractions, that we already know how to integrate. To illustrate the method, observe that by taking the fractions 2/(x – 1) and 1/(x + 2) to a common denominator we obtain

$$\frac{2}{x-1}-\frac{1}{x+2}=\frac{2(x+2)-(x-1)}{(x-1)(x+2)}=\frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this

equation $\int\frac{x+5}{x^2+x-2}\,dx=\int\left(\frac{2}{x-1}-\frac{1}{x+2}\right)dx=-2\ell n\,|x-1|-\ln|x+2|+C$

To see how the method of partial fractions works in general, let's consider a rational

function $f(x)=\frac{P(x)}{Q(x)}$

Where P and Q are polynomials. It's possible to express f as sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called proper. Recall that if $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$

where an ≠ 0, then the degree of P is n and we write deg (P) = n.

If f is improper, that is, deg(P) $\geq$ deg (Q), then we must take the preliminary step of dividing Q into P (by division) until a remainder R(x) is obtained such that deg (R) < deg(Q). The division statement is

(1) $f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$ where S and R are also polynomials. As the following example illustrates, sometimes this preliminary step is all that is required.

Ex.1 Evaluate $\int\frac{x^3+x}{x-1}$ **dx.**

Sol. Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\int \frac{x^3+x}{x-1}dx \int\left(x^2+x+2+\frac{2}{x-1}\right)dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ell n|x-1| + C$$

The next step is to factor the denominator Q(x) as far as possible . It can be shown that any polynomial Q can be factored as a product of linear factors (of the form ax + b) and irreducible quadratic factors

(of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

For instance, if $Q(x) = x^4 - 16$, we could factor it as $Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

The third step is to express the proper rational function R(x)/Q(x) (from equation 1) as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

Case I : The Denominator Q(x) is a product of distinct linear factors.

This means that we can write $Q(x) = (a_1x + b1)(a_2x + b2) \dots (a_kx + bk)$ where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \dots, A_k$ such that.

(2) $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_k}{a_kx+b_k}$$

These constants can be determined as in the following example.

Ex.2 Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}$ **dx.**

Sol. Since the degree of the numerator is less than the degree of the denominator, we don't need to divide.

We factor the denominator as $2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form.

(3) $$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

To denominator the values of A, B and C, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining.

(4) $x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$

Expanding the right side of equation 4 and writing it in the standard form for polynomials, we get

(5) $x^2 + 2x - 1 = (2A + B + 2C) x^2 + (3A + 2B - C) x - 2A$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side-namely, 1. Likewise. The coefficients of x are equal and the constant terms are equal. This gives the following system of equation for A, B and C.

$2A + B + 2C = 1$, $3A + 2B - C = 2$, $-2A = -1$

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int\left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right)dx$$

$$= \frac{1}{2}\ell n\,|x| + \frac{1}{10}\ell n|2x-1| - \frac{1}{10}\ell n|x+2| + K$$

In integrating the middle term we have made the mental substitution $u = 2x - 1$, which gives $du = 2dx$ and $dx = du/2$.

Case II : Q(x) is a product of linear factors, some of which are repeated Suppose the first linear factor $(a_1x + b_1)$ is repeated r times, that is, $(a_1x + b_1)^r$ occurs in the factorization of Q(x). Then instead of the single term $A1/(a_1x + b_1)$ in equation 2, we would use

(6) $$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \dots + \frac{A_r}{(a_1x+b_1)^r}$$

By way of illustration, we could write $\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$

but we prefer to work out in detail a simpler example.

Ex.3 Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$ **dx**

Sol.

The first step is to divide. The result of long division is $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$. Since $Q(1) = 0$, we know that $x - 1$ is a factor and we obtain $x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1) = (x - 1)^2 (x + 1)$ Since the linear factor $x - 1$ occurs twice, the partial fraction decompositoin is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator $(x - 1)^2 (x + 1)$, we get

(7) $4x = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2 = (A + C) x^2 + (B - 2x) x + (-A + B + C)$

Now we equate coefficients : $A + C = 0, B - 2C = 4, -A + B + C = 0$

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$$

$$= \int\left[x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right]dx$$

$$= \frac{x^2}{2} + x + \ell n\,|x-1| - \frac{2}{x-1} - \ell n|x+1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ell n \left|\frac{x-1}{x+1}\right| + K$$

Case III : Q(x) contains irreducible quadratic factors, none of which is repeated If Q(x) has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then in addition to the partial fractions in equation 2 and 6, the expression or R(x)/Q(x) will have a term of the form.

(8) $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined. For instance, the function givenby $f(x) = x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

The term given in (8) can be integrated by completing the square and using the formula.

(9) $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Ex.4 Evaluate $\int \frac{2x^2 - x + 4}{x^3 + 4x}$ **dx**

Sol.

Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write $\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+c}{x^2+4}$

Multiplying by $x(x^2 + 4)$, we have $2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x = (A + B)x^2 + Cx + 4A$ Equating coefficients, we obtain $A + B = 2$ $C = -1$ $4A = 4$

Thus $A = 1$, $B = 1$ and $C = -1$ and so $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx$

In order to integrate the second term we split it into to parts $\int \frac{x-1}{x^2+4} dx = \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2x\,dx$. We evaluate the second integral by means of Formula 9 with $a = 2$.

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}dx = \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

$$= \ell n\,|x| + \frac{1}{2}\,\ell n\,(x^2+4) - \frac{1}{2}\tan^{-1}(x/2) + K$$

Case IV : Q(x) Contains A repeated irreducible quadratic factor. If Q (x) has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction

(10) $\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$

occurs in the partial fraction decomposition of R(x)/Q(x), each of the terms in (10) can be integrated by first completing the square.

Ex.5 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Sol.

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

Ex.6 Evaluate $\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}dx$

Sol. The form of the partial fraction decomposition is $\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$

Multiplying by $x(x^2 + 1)^2$, we have $-x^3 + 2x^2 - x + 1 = A(x^2 + 1)^2 + (Bx + C)\,x\,(x^2 + 1) + (Dx + E)x$

$= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex = (A + B)\,x^4 + Cx^3 + (2A + B + D)\,x^2 + (C + E)\,x + A$

If we equate coefficient, we get the system $A + B = 0$, $C = -1$, $2A + B + D = 2$, $C+E = -1$, $A=1$

Which the solution $A = 1$, $B = -1$, $D = 1$, and $E = 0$. Thus

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = \int\left(\frac{1}{x}-\frac{x+1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx$$

$$= \int\frac{dx}{x} - \int\frac{x}{x^2+1}\,dx - \int\frac{dx}{x^2+1} + \int\frac{x\,dx}{(x^2+1)^2}$$

$$= \ell n\,|x| - \frac{1}{2}\,\ell n\,(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K$$

We note that sometimes partial fractions can be avoided when integrating a rational function. For instance, although the integral $\int \frac{x^2+1}{x(x^2+3)}\,dx$

could be evaluated by the method of case III, it's much easier to observe that if u = $x(x^2+3) = x^3+3x$, then $du = (3x^2+3)\,dx$ and so $\int \frac{x^2+1}{x(x^2+3)}\,dx = \frac{1}{3}\,\ell n\,|x^2+3x| + C$

Ex.7 Evaluate $\int \frac{x^3-3x^2+2x-3}{(x^2+1)^2}$ **dx**

Sol. In this example there is a repeated quadratic polynomial in the denominator. Hence, according to our previous discussion

$\frac{x^3-3x^2+2x-3}{(x^2+1)^2} = \frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{(x^2+1)^2}$ For some constants A_1, B_1, A_2 and B_2

An easy way to determine these constant is as follows. By long division,

$\frac{x^3-3x^2+2x-3}{x^2+1} = x-3+\frac{x}{x^2+1}$ and

therefore $\frac{x^3-3x^2+2x-3}{x^2+1} = \frac{x-3}{x^2+1} + \frac{x}{(x^2+1)^2}$

Thus $A_1 = 1$, $B_1 = -3$, $A_2 = 1$ and $B_2 = 0$

we know have $\int \frac{x^3-3x^2+2x-3}{(x^2+1)^2}\,dx$

$$= \int \frac{x}{x^2+1}\,dx - \int \frac{3}{x^2-1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \frac{1}{2}\ell n\,(x^2+1) - 3\tan^{-1}x - \frac{1}{2(x^2+1)} + C$$

Ex.8 Evaluate $\int \frac{dx}{\cos x + \operatorname{cosec} x}$

Sol.

$$I = \int \frac{dx}{\cos x + \frac{1}{\sin x}} = \int \frac{\sin x\,dx}{\cos x.\sin x + 1}$$

$$= \int \frac{2\sin x\,dx}{2+2\sin x\cos x}dx = \int \frac{2\sin x}{2+\sin 2x}dx$$

$$= \int \frac{[(\sin x+\cos x)+(\sin x-\cos x)]dx}{2+\sin 2x}$$

$$= \int \frac{\sin x+\cos x}{2+\sin 2x}\,dx + \int \frac{\sin x-\cos x}{2+\sin 2x}\,dx$$

$$= \int \frac{\sin x+\cos x}{3-(1-\sin 2x)}\,dx + \int \frac{\sin x-\cos x}{1+(1+\sin 2x)}$$

$$= \int \frac{\sin x+\cos x}{3-(\sin x-\cos x)^2}\,.\,dx + \int \frac{\sin x-\cos x}{1+(\sin x+\cos x)^2}\,.\,dx$$

put $\sin x - \cos x = s$ and $\sin x + \cos x = t \Rightarrow (\cos x + \sin x)\,dx = ds$ and $(\cos x - \sin x)\,dx = dt$

$$I = \int \frac{ds}{3-s^2} - \int \frac{dt}{1+t^2} = \int \frac{ds}{(\sqrt{3})^2-(s)^2} - \int \frac{dt}{1+t^2}$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+s}{\sqrt{3}-s}\right| - \tan^{-1}t + c$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}\right| - \tan^{-1}(\sin x+\cos x) + c$$

Ex.9 Evaluate $\int \frac{1}{(e^x-1)^2}dx$.

Sol.

We have $\int \frac{1}{(e^x-1)^2}dx + \int \frac{e^x}{e^x(e^x-1)^2}dx$, [multiplying the Nr. and Dr. by e^x]

$= \int \frac{dt}{t(t-1)^2}$, putting $e^x = t$ so that $e^x\,dx = dt$.

Now $\frac{1}{t(t-1)} \equiv \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2} \Rightarrow$ $1 \equiv A(t-1)2 + Bt(t-1)+Ct$ ....(1) (on resolving into partial fractions)

To find A, putting t = 0 on both sides of (1), we get A = 1

To find C, put t = 1 and we get C = 1. Thus $1 \equiv (t-1)^2 + Bt(t-1) + t$

Comparing the coefficients of t^2 on both sides, we get 0 = 1 + B or B = – 1

$\therefore \quad \frac{1}{t(t-1)^2} = \frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}$

Hence $\int \frac{dt}{t(t-1)^2} = \int \frac{1}{t}dt - \int \frac{dt}{t-1} + \int \frac{dt}{(t-1)^2} = \log t - \log(t-1) - \{1/(t-1)\} + C$

$= \log e^x - \log(e^x - 1) - \{1/(e^x-1)\} + c = x - \log(e^x-1) - \{1/(e^x-1)\} + c$

Ex.10 Integrate $(3x + 1) / \{(x - 1)^3 (x + 1)\}$.

Sol.

Putting x – 1 = y so that x = 1 + y, we get $\frac{3x+1}{(x-1)^3(x+1)} = \frac{3(1+y)+1}{y^3(2+y)} = \frac{4+3y}{y^3(2+y)}$

arranging the N^r. and the D^r. in ascending powers of y

$= \frac{1}{y^3}\left[2 + \frac{1}{2}y - \frac{1}{4}y^2 + \frac{1}{4}\frac{y^3}{2+y}\right]$, by actual division

$$= \frac{2}{y^3} + \frac{1}{2y^2} - \frac{1}{4y} + \frac{1}{4} \cdot \frac{1}{(2+y)}$$

$$= \frac{2}{(x-1)^3} + \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)}$$

Hence the required integral of the given fraction $= \int \left[\frac{2}{(x-1)^3} + \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)}\right] dx$

$$= \frac{-1}{(x-1)^2} - \frac{1}{2(x-1)} - \frac{1}{4}\log(x-1) + \frac{1}{4}\log(x+1) + c$$

$$= \frac{-1}{(x-1)^2} - \frac{1}{2(x-1)} - \frac{1}{4}\log\frac{x+1}{x-1} + c$$

Ex.11 Evaluate the integral $\int \frac{1}{x^3(x-1)} dx$.

Sol.

Let $x = \sec^2\theta \Rightarrow dx = 2\sec^2\theta \tan q\, d\theta$

$$\Rightarrow \quad I = \int \frac{2\sec^2\theta \tan\theta d\theta}{\sec^6\theta \tan\theta} \Rightarrow 2\int \cos^4\theta d\theta$$

$$I = 2\int \cos^4\theta d\theta = 2\int [(\cos^2 2\theta)^2] d\theta$$

$$= 2\int \left[\frac{1+\cos 2\theta}{2}\right]^2 d\theta = \frac{2}{4}\int (\cos^2 2\theta + 2\cos 2\theta) d\theta$$

$$= \frac{1}{2}\left|\int d\theta + \int \cos^2 2\theta d\theta + 2\int \cos 2\theta d\theta\right|$$

$$= \frac{1}{2}\left|\int d\theta + \int \cos^2 2\theta d\theta + 2\int \cos 2\theta d\theta\right|$$

$$= \frac{1}{2}\left[\theta + \int \left(\frac{1+\cos 4\theta}{2}\right) d\theta + \frac{\sin 2\theta}{2}\right] + c$$

$= \frac{12}{2}\left[\theta + \frac{\theta}{2} + \frac{\sin 4\theta}{8} + \sin 2\theta\right] + c$

$= \frac{\theta}{2} + \frac{\theta}{4} + \frac{\sin 4\theta}{16} + \frac{\sin 2\theta}{2} + c$

$= \frac{3\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{16} + c$ where $x = \sec^2\theta$

Ex.12 Integrate, $\int \frac{2e^{5x} + e^{4x} - 4e^{3x} + 4e^{2x} + 2e^{x}}{(e^{2x} + 4)(e^{2x} - 1)^2} dx.$

Sol.

Put $e^x = y \Rightarrow I = \int \frac{2y^4 + y^3 - 4y^2 + 4y + 2}{(y^2 + 4)(y^2 - 1)^2} dy$

$= \int \frac{y(y^2 + 4)(y^4 - 2y^2 + 1)}{(y^2 + 4)(y^2 - 1)^2} dy$

$= -\frac{1}{2(e^{2x} - 1)} + \tan^{-1}\left(\frac{e^x}{2}\right) + c$

Ex.65 Integrate $\int \frac{dy}{y^2(1 + y^2)^3}$.

Sol.

Put $y = \tan\theta$

$\Rightarrow \int \frac{dy}{y^2(1 + y^2)^3} = \int \frac{\cos^6\theta}{\sin^2\theta} d\theta = \int \frac{(1 - \sin^2\theta)^3 \, d\theta}{\sin^2\theta}$

$= \int \frac{(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)d\theta}{\sin^2\theta}$

$= \int (\mathrm{cosec}^2\theta - 3 + 3\sin^2\theta - \sin^4\theta)\, d\theta$

$= -\frac{1}{y} - \frac{15}{8}\tan^{-1} y - \frac{1}{2}\sin(2\tan^{-1}y) - \frac{1}{32}\sin(4\tan^{-1}y) + c$

Ex.13 Evaluate $\int \frac{f(x)}{x^3 - 1}dx$, where f(x) is a polynomials of degree 2 in x such that $f(0) = f(1) = 3f(2) = -3$.

Sol.

Let , $f(x) = ax^2 + bx + c$ given, $f(0) = f(1) = 3f(2) = -3$

$\therefore f(0) = f(1) = 3f(2) = -3$, $f(0) = c = -3$, $f(1) = a + b + c = -3$, $3f(2) = 3(4a + b + c) = -3$

on solving we get $a = 1, b = -1, c = -3$

$\therefore f(x) = x^2 - x - 3$

$\Rightarrow I = \int \frac{f(x)}{x^3 - 1}dx = \int \frac{x^2 - x - 3}{(x-1)(x^2 + x + 1)}dx$

Using partical fractions, we get, $\frac{(x^2 - x - 3)}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2 + x + 1)}$

We get, $A = -1, B = 2, C = 2$

$\therefore \quad I = \int -\frac{1}{x-1}dx + \int \frac{(2x+2)}{(x^2+x+1)}dx$

$= -\log|x-1| + \int \frac{(2x+2)}{(x^2+x+1)} + \int \frac{1-dx}{x^2+x+1}$

$= -\log|x-1| + \log|x^2+x+1| + \int \frac{dx}{(x+1/2)^2 + (\sqrt{3}/2)^2}$

$= \log|x-1| + \log|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

Ex.14 Integrate 1/(sinx + sin 2x).

Sol.

We have $I = \int \frac{dx}{\sin x + \sin 2x}$

$$= \int \frac{dx}{\sin x + 2\sin x \cos x} = \int \frac{dx}{\sin x(1 + 2\cos x)}$$

$$= \int \frac{\sin x \, dx}{\sin^2(1 + 2\cos x)} = \int \frac{\sin x dx}{(1 - \cos^2 x)(1 + 2\cos x)}$$

Now putting $\cos x = t$, so that $-\sin x \, dx = dt$, we get

$$I = -\int \frac{dt}{(1 - t^2)(1 + 2t)} = -\int \frac{dt}{(1 - t)(1 + t)(1 + 2t)}$$

$$= -\int \left[\frac{1}{6(1 - t)} - \frac{1}{2(1 - t)} + \frac{4}{3(1 + 2t)} \right] dt,$$

$$= \frac{1}{6} \log(1-t) + \frac{1}{2} \log(1+t) - \frac{2}{3} \log(1+2t) + c$$

$$= \frac{1}{6} \log(1 - \cos x) + \frac{1}{2} \log(1+\cos x) - \frac{2}{3} \log(1+2\cos x) + c$$

Integrals of Some Particular Functions

Introduction

Consider the integral $\int \sqrt{a^2 - x^2}$ dx

If we change the variable from x to θ by the substitution $x = a \sin\theta$, then the identity

$$1 - \sin^2\theta = \cos^2\theta$$

allows us to get rid of the roots sign because

$\int \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a|\cos\theta|$ Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) are the substitution $x = a \sin\theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

$$\int f(x)dx = \int f(g(t))g'(t)dt$$

This kind of substitution is called **inverse substitution.**
We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function. This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

Some Standard Integrals

(i) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$

(ii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(iii) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$

(iv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln [x + \sqrt{x^2 + a^2}]$

(v) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln [x + \sqrt{x^2 - a^2}]$

(vi) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left|\frac{a + x}{a - x}\right| + c$

(vii) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left|\frac{x - a}{x + a}\right| + c$

(viii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

(ix) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left|x + \sqrt{x^2 + a^2}\right| + c$

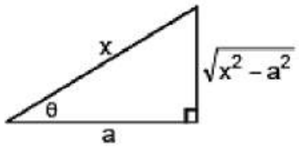
(x) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left|x + \sqrt{x^2 - a^2}\right| + c$

Solved Examples

Ex.1 Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, **where a > 0**

Sol.

We let $x = a \sec\theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$. Then $dx = a \sec\theta \tan\theta \, d\theta$ and



$\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$

$= \sqrt{a^2 \tan^2\theta} = a\,|\tan\theta| = a \tan\theta$

Therefore $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec\theta \tan\theta}{a \tan\theta}\, d\theta$

$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c$

The triangle in figure gives $\tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$,

so we have $\frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + c$

$= \ln|x + \sqrt{x^2 - a^2}| - \ln a + C$

$= \ln|x + \sqrt{x^2 - a^2}| + C$, $\sec\theta = \frac{x}{a}$

Ex.2 Integrate $1/(2x^2 + x - 1)$.

Sol.

We have $\int \frac{dx}{(2x^2 + x - 1)} = \frac{1}{2} \int \frac{dx}{\left(x^2 + \frac{x}{2} - \frac{1}{2}\right)}$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2 - \frac{1}{2} - \frac{1}{16}} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2 - \frac{9}{16}}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\frac{4}{3}\log \frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}} + C$$

$$= \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + c = \frac{1}{3}\log\left|\frac{2x-1}{x+1}\right| - \frac{1}{3}\log 2 + c$$

$$= \frac{1}{3}\log (2x-1)/(x-1)| + C_1$$

Ex.3 Integrate (3x + 1) / (2x² – 2x + 3).

Sol. Here $(d/dx)(2x^2 - 2x + 3) = 4x - 2$.

$$\therefore \quad I = \int \frac{3x+1}{2x^2-2x+3}\,dx = \int \frac{\frac{3}{4}(4x-2)+1+\frac{3}{2}}{(2x^2-2x+3)}dx$$

$$= \frac{3}{4}\int \frac{4x-2}{2x^2-2x+3}\,dx + \frac{5}{2}\int \frac{1}{2x^2-2x+3}\,dx$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{2.2}\int \frac{dx}{x^2-x+(3/2)}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right) - \left(\frac{1}{4}\right)}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + (\sqrt{5/2})^2}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\,\frac{1}{(\sqrt{5/2})}\left(\tan^{-1}\left\{\frac{x-\frac{1}{2}}{(\sqrt{5/2})}\right\}\right) + c$$

Ex.4 Integrate $1/\sqrt{(4+3x-2x^2)}$.

Sol.

We have $\int \frac{dx}{\sqrt{(4+3x-2x^2)}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left\{2+\frac{9}{16}-\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)\right\}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{(41/16)-(x-3/4)^2\}}}$

$= \frac{1}{\sqrt{2}} \sin^{-1}\left\{\frac{x-\frac{3}{4}}{(\sqrt{41/4})}\right\} + c$

$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4x-3}{41}\right) + c$

Ex.5 Evaluate $\int \sqrt{(x-1)(2-x)}$ **dx.**

Sol.

We have $\int \sqrt{(x-1)(2-x)}\ dx = \int \sqrt{(-x^2+3x-2)}dx$

$= \int \sqrt{\left\{-2-\left(x-\frac{3}{2}\right)^2+\frac{9}{4}\right\}} dx \int \sqrt{\left\{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2\right\}} dx,$

$= \frac{1}{2}\left(x-\frac{3}{2}\right)\sqrt{\left\{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2\right\}} + \frac{1}{2}\cdot\frac{1}{4}\sin^{-1}\left\{\left(x-\frac{3}{2}\right)/(1/2)\right\} + c$

$= \frac{1}{4}(2x-3)\sqrt{(3x-x^2-2)} + \frac{1}{8}\sin^{-1}(2x-3) + c$

Ex.6 Integrate $\int \frac{(x^3+3)dx}{\sqrt{(x^2+1)}}$

Sol.

We have $\int \frac{(x^3+3)dx}{\sqrt{(x^2+1)}} = \int \frac{x(x^2+1)-x+3}{\sqrt{(x^2+1)}}\,dx$

$= \int \frac{x(x^2+1)}{\sqrt{(x^2+1)}}\,dx - \int \frac{xdx}{\sqrt{(x^2+1)}} + 3\int \frac{dx}{\sqrt{(x^2+1)}}$

$= \frac{1}{2}\int (2x)\sqrt{(x^2+1)}\,dx - \frac{1}{2}\int \frac{2x\,dx}{\sqrt{(x^2+1)}} + 3\int \frac{dx}{\sqrt{(x^2+1)}}$

$= \frac{1}{2}\left[\frac{2}{3}(x^2+1)^{3/2}\right] - \frac{1}{2}\,[2\sqrt{(x^2+1)}\,] + 3\,\ell n\,(x+\sqrt{(x^2+1)}\,) + c$

$= \frac{1}{3}\,(x^2+1)^{2/3} - \sqrt{(x^2+1)} + 3\,\ell n\,(x+\sqrt{(x^2+1)}\,) + c$

Ex.7 Integrate $x^2/(x^4+x^2+1)$

Sol.

Let $I = \int \frac{x^2}{x^4+x^2+1}\,dx, = \int \frac{1}{x^2+1+\frac{1}{x^2}}\,dx,$ dividing the numerator and the denominator both by x^2.

Now the denominator x^2+1+1/x^2 can be written either as $\left(x-\frac{1}{x}\right)^2+1$ or as $\left(x+\frac{1}{x}\right)^2-1$. The diff. coeff. of $x-1/x$ is $1+1/x^2$ and that of $x+1/x$ is $1-1/x^2$. So we write

$I = \frac{1}{2}\int \frac{(1+1/x^2)+(1-1/x^2)}{x^2+1+(1/x^2)}\,dx$

$= \frac{1}{2}\int \frac{(1+1/x^2)dx}{(x-1/x)^2+3} + \frac{1}{2}\int \frac{(1-1/x^2)dx}{(x+1/x)^2-1}$

In the first integral put $x-1/x = t$ so that $\left(1+\frac{1}{x^2}\right)dx = dt,$ and in the second integral put

$x + \frac{1}{x} = z$ so that $\left(1 - \frac{1}{x^2}\right) dx = dz.$

$\therefore \quad I = \frac{1}{2} \int \frac{dt}{t^2 + (\sqrt{3})^2} + \frac{1}{2} \int \frac{dz}{z^2 - 1}$

$= \frac{1}{2\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + \frac{1}{2} \frac{1}{2 \times 1} \log \frac{z-1}{z+1} + c$

$\frac{1}{2\sqrt{3}} \tan^{-1} \left\{\frac{(x - 1/x)}{\sqrt{3}}\right\} + \frac{1}{4} \log \frac{(x + 1/x) - 1}{(x + 1/x) + 1} + c$

$= \frac{1}{2\sqrt{3}} \tan^{-1} \left\{\frac{x^2 - 1}{(\sqrt{3})x}\right\} + \frac{1}{4} \log \frac{x^2 - x + 1}{x^2 + x + 1} + c$

Ex.8 Evaluate $\int \frac{(1 + x^2)dx}{(1 - x^2)\sqrt{1 + x^2 + x^4}}$

Sol.

Let, $I = \int \frac{(1 + x^2)dx}{(1 - x^2)\sqrt{1 + x^2 + x^4}}$

$= \int \frac{x^2\left(1 + \frac{1}{x^2}\right)dx}{x^2\left(\frac{1}{x} - x\right)\sqrt{\frac{1}{x^2} + 1 + x^2}}$

$= - \int \frac{(1 + 1/x^2)dx}{(x - 1/x)\sqrt{(x - 1/x)^2 + 3}}$

$= - \int \frac{dt}{t\sqrt{t^2 + 3}}$ (put $x - \frac{1}{x} = t$)

Again put $t^2 + 3 = s^2$

$\Rightarrow \quad 2t\,dt = 2s\,ds = - \int \frac{s\,ds}{s(s^2 - 3)}$

$$= -\int \frac{ds}{s^2 - (\sqrt{3})^2} = -\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{(x-1/x)^2+3} - \sqrt{3}}{\sqrt{(x-1/x)^2+3} + \sqrt{3}} \right| + c$$

$$= -\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{x^2 + \frac{1}{x^2} + 1} - \sqrt{3}}{\sqrt{x^2 + \frac{1}{x^2} + 1} + \sqrt{3}} \right| + c$$

Ex.9 Evaluate $\int \frac{(x-1)dx}{(x+1)\sqrt{x^3 + x^2 + x}}$

Sol.

Let, $I = \int \frac{(x-1)dx}{(x+1)\sqrt{x^3+x^2+x}} = \int \frac{(x^2-1)dx}{(x+1)^2\sqrt{x^3+x^2+x}}$

$$= \int \frac{x^2(1-1/x^2)dx}{(x^2+2x+1)\sqrt{x^3+x^2+x}}$$

$$= \int \frac{x^2\left(1-\frac{1}{x^2}\right)dx}{x^2\left(x+2+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}} = \int \frac{dt}{(t+2)\sqrt{t+1}}$$

(put $x + \frac{1}{x} = t$, $(1 - 1/x^2)\,dx = dt$)

$$= \int \frac{2z\,dz}{(z^2+1)z} = 2\int \frac{dz}{z^2+1} = 2\tan^{-1}(z) + c$$

(put $t + 1 = z^2 \Rightarrow dt = 2zdz$)

$$= 2\tan^{-1}(\sqrt{t+1}) + c = 2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + c$$

Ex.10 Evaluate $\int \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}$

Sol.

Put $x = \alpha\cos 2\theta + \beta\sin 2\theta$ so that $dx = 2(\beta - \alpha)\sin\theta\cos\theta\,d\theta$

Also $(x - \alpha) = (\beta - \alpha)\sin 2\theta$, and $(\beta - x) = (\beta - \alpha)\cos 2\theta$

Making these substitutions,in the given

integral $= \int \frac{2(\beta-\alpha)\sin\theta\cos\theta d\theta}{\sqrt{\{(\beta-\alpha)\cos^2\theta.(\beta-\alpha)\sin^2\theta\}}} = \int \frac{2(\beta-\alpha)\sin\theta\cos\theta}{(\beta-\alpha)\cos\theta\sin\theta} d\theta$

$= 2\int d\theta = 2\theta = \cos^{-1}(\cos 2\theta)$ ....(1)

But $x = \alpha\cos 2\theta + \beta\sin 2\theta$; $2x = \alpha(1 + \cos 2\theta) + \beta(1 - \cos 2\theta)$

i.e., $(\beta - \alpha)\cos 2\theta = (\alpha + \beta - 2x)$ or $\cos 2\theta = (\alpha + \beta - 2x) / (\beta - \alpha)$

$\therefore$ from (1), we get the given integral $= \cos^{-1}\left(\frac{\alpha+\beta-2x}{\beta-\alpha}\right)$.

Ex.11 Evaluate $I = \int \frac{dx}{(a+dx^2)\sqrt{b-ax^2}}$

Sol.

Substituting $ax^2 = b\sin^2\theta \Rightarrow$ $dx = \sqrt{\frac{b}{a}}\cos\theta\, d\theta$

$\therefore I = \int \frac{\sqrt{\frac{b}{a}}\cos\theta d\theta}{\left(a + \frac{b^2}{a}\sin^2\theta\right)\sqrt{b - b\sin^2\theta}}$

$= \sqrt{a}\int \frac{\cos\theta d\theta}{(a^2 + b^2\sin^2\theta).\cos\theta} = \sqrt{a}\int \frac{d\theta}{a^2 + b^2\sin^2\theta}$, dividing N^r and D^r by $\cos 2\theta$. we get

$= \sqrt{a}\int \frac{\sec^2\theta d\theta}{a^2\sec^2\theta + b^2\tan^2\theta}$ put $\tan\theta = t$

$= \sqrt{a}\int \frac{dt}{a^2(1+t^2) + b^2t^2} = \frac{\sqrt{a}}{(a^2+b^2)}\int \frac{dt}{t^2 + \frac{a^2}{a^2+b^2}}$

$= \frac{1}{\sqrt{a(a^2+b^2)}}\tan^{-1}\left(\frac{t\sqrt{a^2+b^2}}{a}\right) + c$

$$= \frac{1}{\sqrt{a(a^2+b^2)}} \cdot \tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right) + c$$

(since, $t = \tan\theta = \frac{x}{\sqrt{b-ax^2}}$)

Ex.12 Integrate $1/(1 + 3\sin^2 x)$.

Sol. Dividing N^r. and D^r. by $\cos^2 x$, we have

$$I = \int \frac{dx}{1+3\sin^2 x} = \int \frac{\sec^2 x dx}{\sec^2 x + 3\tan^2 x}$$

$$= \int \frac{\sec^2 x dx}{(1+\tan^2 x)+3\tan^2 x} = \int \frac{\sec^2 x dx}{1+\tan^2 x}$$

Now putting $2 \tan x = 5$ so that $2 \sec^2 x \, dx = dt$, we have

$$I = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1}(2\tan x)$$

Integration by Parts

Definition

$\int u.v dx = u \int v dx - \int \left[\frac{du}{dx}.\int v dx\right] dx$ where u & v are differentiable functions.

Note : While using integration by parts, choose u & v such that

(a) $\int$ dx v is simple &

(b) $\int \left[\frac{du}{dx}.\int v \, dx\right]$ dx is simple to integrate.

This is generally obtained, by keeping the order of u & v as per the order of the letter in **ILATE**, where

I – Inverse function
L – Logarithmic function
A – Algebraic function

T – Trigonometric function
E – Exponential function

Remember This:

(i) $\int e^{ax} . \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$

(ii) $\int e^{ax} . \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$

Evaluate $I = \int e^{ax} \sin bx dx$

Proof:

Integrating by parts taking sin bx as the second function,

We get $I = -\frac{e^{ax} \cos bx}{b} - \int a e^{ax}\left(-\frac{\cos bx}{b}\right) dx = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \int e^{ax} \cos bx dx$

Again integrating by parts taking cos bx as the second function, we get

$$I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b}\left[\frac{e^{ax} \sin bx}{b} - \int a e^{ax} \frac{\sin bx}{b} dx\right]$$

or $I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx dx$

or $I = \frac{e^{ax}}{b^2}(a \sin bx - b \cos bx) - \frac{a^2}{b^2} I.$

Transposing the term $-a^2/b^2$ I to the left hand side, we get $\left(1+\frac{a^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a \sin bx - b \cos bx)$

or $\frac{1}{b^2}(a^2+b^2) I = \frac{1}{b^2} e^{ax}(a \sin bx - b \cos bx)$

$\therefore I = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

Thus, $\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$

Remark : (i) $\int e^{x}[f(x) + f'(x)]\,dx = e^{x}.f(x) + c$

(ii) $\int [f(x) + xf'(x)]dx = xf(x) + c$

Solved Examples

Ex.1 Integrate $x^n \log x$

Sol.

We have $\int x^n \log x\,dx = \int (\log x).x^n dx$

$= (\log x) . \frac{x^{n+1}}{n+1} - \int \frac{1}{x}.\frac{x^{n+1}}{n+1}\,dx$

$= (\log x) . \frac{x^{n+1}}{n+1} - \int \frac{x^n}{n+1}dx$

$= (\log x) . \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$

Ex.2 Evaluate $\int \frac{\log(\sec^{-1} x)dx}{x\sqrt{(x^2-1)}}$ **dx**

Sol.

Put $\sec^{-1} x = t$ so that so that $\frac{1}{x\sqrt{(x^2-1)}}\,dx = dt.$

Then the given integral $= \int \log t\,dt = \int (\log t).1\,dt$

$= (\log t) . t - \int \frac{1}{t} t\,dt = t \log t - t + c$

$= t(\log t - \log e) + c = \sec^{-1} x (\log(\sec^{-1} x) - 1) + c$ $= \sec^{-1} x \left(\log\left(\frac{\sec^{-1} x}{c}\right)\right) + c$

Ex.3 Evaluate $\int \tan^{-1} \sqrt{\left(\frac{1-x}{1+x}\right)} dx$.

Sol.

Put $x = \cos\theta$ so that $dx = -\sin\theta\, d\theta$. the given integral

$$= \int \left\{\tan^{-1} \sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}\right\} (-\sin\theta) d\theta$$

$$= -\int (\tan^{-1}(\tan \frac{\theta}{2}) \sin\theta\, d\theta)$$

$$= -\int \frac{\theta}{2} \sin\theta\, d\theta = -\frac{1}{2} \int \theta \sin\theta\, d\theta$$

$$= -\frac{1}{2} [\theta . (-\cos\theta) - \int (-\cos\theta) d\theta]$$

$$= \frac{\theta\cos\theta}{2} - \frac{\sin\theta}{2} = \frac{1}{2} [x \cos^{-1} x - \sqrt{(1-x^2)}]$$

Ex.4 Evaluate $\int x^2 \tan^{-1} x\, dx$.

Sol.

We have $\int x^2 \tan^{-1} x\, dx = \frac{x^3}{3} \tan^{-1} x - \int \frac{x^3}{3} . \frac{1}{1+x^2}$,

$$\frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x(x^2+1) - x}{1+x^2} dx$$ $[x^3 = x(x^2 + 1) - x]$

integrating by parts taking x^2 as the second function

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x\, dx + \int \frac{1}{3} . \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\log(1+x^2) + c$

Ex.5 Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)$**dx.**

Sol.

$$I = \int \sin^{-1}\left\{\frac{2x+2}{\sqrt{4x^2+8x+13}}\right\}dx = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right)dx$$

$$= \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\, d\theta = \frac{3}{2}\int \theta\sec^2\theta d\theta$$ (put, $2x + 2 = 3\tan\theta \Rightarrow 2\,dx = 3\sec2\theta\, d\theta$)

$$= \frac{3}{2}(\theta\tan\theta - \int \tan\theta d\theta) = \frac{3}{2}\{\theta\tan\theta - \log(\sec\theta)\} + c$$

$$I = \frac{3}{2}\left\{\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\left(\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right)\right\} + C$$

$$= \frac{3}{2}\left\{\frac{2}{3}(x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \log\frac{\sqrt{4x^2+8x+13}}{3}\right\} + c$$

$$\Rightarrow \quad I = (x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \frac{1}{4}\log(4x^2+8x+13) + c$$

Ex.6 If $\cos\theta > \sin\theta > 0$, then evaluate : $\int\left\{\log\left(\frac{1+\sin2\theta}{1-\sin2\theta}\right)^{\cos^2\theta} + \log\left(\frac{\cos2\theta}{1+\sin2\theta}\right)\right\}d\theta$

Sol.

$$\text{Here, } I = \int\left\{\log\left(\frac{1+\sin2\theta}{1-\sin2\theta}\right)^{\cos^2\theta} + \log\left(\frac{\cos2\theta}{1+\sin2\theta}\right)\right\}d\theta$$

$$= \int\left\{2\cos^2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right\}d\theta$$

$$= \int (2\cos^2\theta - 1)\log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)d\theta$$

$$= \int \underset{II}{\cos 2\theta}.\underset{I}{\log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)}d\theta$$, applying integration by parts

$$= \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right).\frac{\sin 2\theta}{2} - \int \frac{2}{\cos 2\theta} \cdot \frac{\sin 2\theta}{2}d\theta$$

$$= \frac{\sin 2\theta}{2}\log\left|\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right| + \frac{1}{2}\log|\cos 2\theta| + c$$

Ex.7 Evaluate $\int \frac{xe^x}{(x+1)^2}dx$

Sol.

We have $\int \frac{xe^x}{(x+1)^2}dx = \int xe^x \frac{1}{(x+1)^2}dx$

$$\int \frac{xe^x}{(x+1)^2}dx = (xe^x)\left(-\frac{1}{x+1}\right) - \int (e^x + xe^x)\left(-\frac{1}{x+1}\right)dx,$$

[Note that the integral of $\frac{1}{(x+1)^2}$ is $-\frac{1}{x+1}$]

$$= -\frac{xe^x}{(x+1)^2} + \int e^x(x+1)\frac{1}{x+1}dx$$

$$= -\frac{xe^x}{x+1} + \int e^x dx = -\frac{xe^x}{x+1} + e^x + c$$

$$= e^x\left[1 - \frac{x}{x+1}\right] + c = e^x\frac{x+1-x}{x+1} + c = \frac{e^x}{x+1} + c$$

Alternative solution

We have $\int \frac{xe^x}{(x+1)^2}dx = \int e^x\frac{(x+1)-1}{(x+1)}dx$

$= \int e^x \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] dx$

$= \int e^x [f(x) + f'(x)\,dx$, where $f(x) = \frac{1}{x+1}$ $= e^x \frac{1}{x+1} + c$

Ex.8 Evaluate $\int e^x \frac{2+\sin 2x}{1+\cos 2x} dx$

Sol.

We have $\int e^x \frac{2+\sin 2x}{1+\cos 2x} dx$

$= \int e^x \left[\frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x}\right] dx$

$= \int e^x [\sec^2 x + \tan x] dx$

$= \int e^x [f(x) + f'(x)] dx$, x where $f(x) = \tan x = e^x f(x) + c = e^x \tan x + c$

Ex.9 Evaluate $\int \frac{dx}{(x^2+a^2)^3}$ ....(i)

Sol.

$I_1 = \int \frac{1}{(x^2+a^2)^2} dx$ ....(ii)

$= \int \underset{\text{I}}{\frac{1}{(x^2+a^2)^2}} . \underset{\text{II}}{1dx} = \frac{1}{(x^2+a^2)^2} . x - \int \frac{-2(2x)}{(x^2+a^2)^3} x dx$

$= \frac{x}{(x^2+a^2)^2} + 4 \int \frac{x^2+a^2-a^2}{(x^2+a^2)^3} dx$

$= \frac{x}{(x^2+a^2)^2} + 4 \int \frac{1}{(x^2+a^2)^2} dx - 4a^2 \int \frac{dx}{(x^2+a^2)^3}$

using, previous example $\Rightarrow I_1 = \frac{x}{(x^2+a^2)^2} + 4I_1 - 4a^2 . I$ (using (i) and (ii))

$$\Rightarrow 4a^2 = \frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2} I_1 \quad \text{....(iii)}$$

$$I_1 = \int \frac{dx}{(x^2+a^2)^2} = \int \frac{x}{2a^2(x^2+a^2)^2} + \frac{1}{2a^3} \tan^{-1}\left(\frac{x}{a}\right) + c$$

Integration of Irrational Functions

Definition

Certain types of integrals of algebraic irrational expressions can be reduced to integrals of rational functions by a appropriate change of the variable. Such transformation of an integral is called its rationalization.

1. If the integrand is a rational function of fractional powers of an independent variable x, i.e. the function $R\left(x, x^{\frac{p_1}{q_1}}, \ldots, x^{\frac{p_k}{q_k}}\right)$, then the integral can be rationalized by the substitution $x = t^m$, where m is the least common multiple of the numbers $q_1, q_2, \ldots, q_k$.
2. If the integrand is a rational function of x and fractional powers of a linear fractional function of the form $\frac{ax+b}{cx+d}$, then rationalization of the integral is effected by the substitution $\frac{ax+b}{cx+d} = t^m$ where m has the same sense as above.

Solved Examples

Ex.1 Evaluate $\int \frac{dx}{\sqrt{(x+a)} + \sqrt{(x+b)}}$.

Sol.

Rationalizing the denominator, we have $\int \frac{dx}{\sqrt{(x+a)} + \sqrt{(x+b)}} \int \frac{\sqrt{(x+b)} - \sqrt{(x+a)}}{(x+a)-(x+a)} dx$

$$= \int \frac{(x+b)^{1/2} - (x+a)^{1/2}}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{2}{3}(x+b)^{3/2} - \frac{2}{3}(x+a)^{3/2}\right]$$

$$= \frac{2}{3}\,\frac{1}{(b-a)}\,[(x+b)^{3/2} - (x+a)^{3/2}] + c$$

Ex.2 Evaluate $I = \int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1+\sqrt[3]{x})}$.

Sol. The least common multiple of the numbers 3 and 6 is 6, therefore we make the substitution

$x = t^6$, $dx = 6t^5\, dt$.

whence $I = 6\int \frac{(t^6 + t^4 + t)t^5}{t^6(1+t^2)}\, dt = 6\int \frac{t^5 + t^3 + 1}{1+t^2}\, dt$

$$= 6\int t^3 dt + 6\int \frac{dt}{t^2+1} = \frac{3}{2}\, t^4 + 6 \arctan t + C.$$

Ex.3 Evaluate $I = \int \frac{(2x-3)^{1/2} dx}{(2x-3)^{1/3} + 1}$.

Sol. The integrand is a rational function of $\sqrt[6]{2x-3}$ therefore we put $2x - 3 = t^6$, whence

$$I = \int \frac{3t^8}{t^2+1}\, dt = 3\int (t^6 - t^4 + t^2 - 1)\, dt + 3\int \frac{dt}{1+t^2}$$

$$= 3\,\frac{t^7}{7} - 3\,\frac{t^5}{5} + 3\,\frac{t^3}{3} - 3t + 3 \arctan t + C.$$

Returning to x, we get

$$I = 3\left[\frac{1}{7}(2x-3)^{7/6} - \frac{1}{5}(2x-3)^{5/6}\frac{1}{3}(2x-3)^{1/2} - (2x-3)^{1/6} + \arctan(2x-3)^{1/6}\right] + C.$$

Ex.4 Evaluate $\int \sqrt[3]{x}\ \sqrt[7]{1+\sqrt[3]{x^4}}\ dx.$

Sol.

Let $x = t^3 \Rightarrow dx = 3t^2$ then

$$I = \int t(1+t^4)^{1/7} \cdot 3t^2\, dt = 3\int t^3(1+t^4)^{1/7} dt$$

Let $1 + t^4 = X^7 \Rightarrow 4t^3\, dt = 7X^6\, dX$

$$= \frac{3}{4} \cdot \int 7X^7 dX = \frac{21}{32} X^6 + C.$$

Therefore $I = \dfrac{21}{32}\ (1 + x^{4/3})^{8/7} + C$

Ex.5 Evaluate $I = \displaystyle\int \frac{2}{(2-x)^2}\ \sqrt[3]{\frac{2-x}{2+x}}\ dx.$

Sol. The integrand is a rational function of x and the expression $\sqrt[3]{\dfrac{2-x}{2+x}}$, therefore let us introduce the substitution

$\sqrt[3]{\dfrac{2-x}{2+x}} = t;\ \dfrac{2-x}{2+x} = t^3$, Whence

$$x = \frac{2-2t^3}{1+t^3};\ 2-x = \frac{4t^3}{1+t^3};\ dx = \frac{-12t^2}{(1+t^3)^2}\ dt.$$

Hence $I = -\displaystyle\int \frac{2(1+t^3)^2\ t\,.1.\,2t^2}{16t^6(1+t^3)^2}\ dt = -\frac{3}{2}\int \frac{dt}{t^3}$

$= \dfrac{3}{4t^2} + C$. We get $I = \dfrac{3}{4}\sqrt[3]{\left(\dfrac{2+x}{2-x}\right)^2} + C.$

Integrals of the type $\int \frac{dx}{x\sqrt{y}}$ where x & y are linear or quadratic expressions

Ex.1 Integrate $1/[(2x + 1)\sqrt{4x + 3}\,]$.

Sol.

Put $4x + 3 = t^2$, so that $4dx = 2tdt$ and $(2x + 1) = \frac{2(t^2-3)}{4} + 1 = \frac{t^2-3}{2} + 1 = \frac{t^2-1}{2}$

$$\int \frac{dx}{[(2x+1)\sqrt{(4x+3)}]} = \int \frac{\frac{1}{2}tdt}{\frac{1}{2}(t^2-1)t}$$

$$= \int \frac{dt}{(t^2-1)} = \frac{1}{2}\log\left\{\frac{t-1}{t+1}\right\} = \frac{1}{2}\log\frac{\sqrt{(4x+3)}-1}{\sqrt{(4x+3)}+1} + c.$$

Ex.2 Evaluate $\int \frac{x^2dx}{(x-1)\sqrt{(x+2)}}$.

Sol.

Put $(x + 2) = t^2$, so that $dx = 2t\,dt$, Also $x = t^2 - 2$.

$$\therefore \int \frac{x^2dx}{(x-1)\sqrt{(x+2)}} = \int \frac{(t^2-2)^2.2t\,dt}{(t^2-3).t} = 2\int \frac{t^4-4t^2+4}{t^2-3}dt$$

$= 2\int [t^2 - 1) + \{1/(t^2-3)\}]\,dt,$ dividing the numerator by the denominator

$$= 2\left[\frac{1}{3}t^3 - t + \{1/(2\sqrt{3})\}\log\{(t-\sqrt{3})/(t+\sqrt{3})\}\right]$$

$$= 2\left[\frac{(x+2)^{3/2}}{3} - \sqrt{(x+2)} + \frac{1}{2\sqrt{3}}\log\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{(x+2)}+\sqrt{3}}\right] + c.$$

Ex.3 Integrate $1/\{x^2\sqrt{(x+1)}\,\}$.

Sol.

Put $(x + 1) = t^2$, so that $dx = 2t\,dt$. Also $x = t^2 - 1$.

$$\int \frac{dx}{x^2\sqrt{(x+1)}} = \int \frac{2t\,dt}{(t^2-1)^2.t} = 2\int \frac{dt}{(t+1)^2(t-1)^2}.$$

$$= \int \frac{1}{2}\left[\frac{1}{(t+1)^2} + \frac{1}{(t+1)} + \frac{1}{(t-1)^2} - \frac{1}{(t-1)}\right]dt,$$

$$= \frac{1}{2}\int \frac{dt}{(t+1)^2} + \frac{1}{2}\int \frac{dt}{(t+1)} + \frac{1}{2}\int \frac{dt}{(t-1)^2} - \frac{1}{2}\int \frac{dt}{(t-1)}$$

$$- \frac{1}{2}\{1/(t+1)\} + \frac{1}{2}\log(t+1) - \frac{1}{2}\{1/(t-1)\} - \frac{1}{2}\log(t-1) + c$$

$$= -\frac{1}{2}\{1/(t+1)\} + 1/(t-1)] + \frac{1}{2}\log\{(t+1)/(t-1)\} + c$$

$$= -\frac{\sqrt{(x+1)}}{x} + \frac{1}{2}\log\left\{\frac{\sqrt{(x+1)}+1}{\sqrt{(x+1)}-1}\right\} + c$$

Ex.4 Integrate $1/[(1+x)\sqrt{(1-x^2)}]$.

Sol.

Put $(1 + x) = 1/t$, so that $dx = -(1/t^2)\,dx$.

Also $x = (1/t) - 1$.

$$\int \frac{dx}{(1+x)\sqrt{(1-x^2)}} = \int \frac{-(1/t^2)dt}{(1/t)\sqrt{[1-\{(1/t)-1\}^2]}}$$

$$= -\int \frac{dt}{\sqrt{[t^2-(1-t)^2]}} = -\int \frac{dt}{\sqrt{(2t-1)}}$$

$$= -\frac{1}{2}\int (2t-1)^{-1/2}.\ (2dt = -\sqrt{(2t-1)})$$

$$= -\sqrt{\left[\frac{2}{1+x}-1\right]} + c = -\sqrt{\left(\frac{1-x}{1+x}\right)} + c.$$

Ex.5 Evaluate $\int \frac{dx}{(x^2-1)\sqrt{(x^2+1)}}$.

Sol.

Put $x = 1/t$, so that $dx = -(1/t^2)\,dt$.

$$\therefore \quad I = \int \frac{dx}{(x^2-1)\sqrt{(x^2+1)}} = \int \frac{-(1/t^2)dt}{(1/t^2)\sqrt{\{(1/t^2)+1\}}} = -\int \frac{t\,dt}{(1-t^2)\sqrt{(1+t^2)}}$$

Now put $1 + t^2 = z^2$ so that $t\,dt = z\,dz$. Then

$$I = -\int \frac{d\,dz}{[1-(z^2-1)]z} = \int \frac{dz}{2-z^2} = \int \frac{dz}{z^2-2}.$$

$$= \frac{1}{2\sqrt{2}} \log \frac{z-\sqrt{2}}{z+\sqrt{2}} = \frac{1}{2\sqrt{2}} \log \frac{\sqrt{(1+t^2)}-\sqrt{2}}{\sqrt{(1+t^2)}+\sqrt{2}} + c$$

$$= \frac{1}{2\sqrt{2}} \log \frac{\sqrt{\{1+(1/x^2)\}}-\sqrt{2}}{\sqrt{\{1+(1/x^2)\}}+\sqrt{2}} + c$$

$$= \frac{1}{2\sqrt{2}} \log \left[\frac{\sqrt{1+(1/x^2)}-x\sqrt{2}}{\sqrt{1+(1/x^2)}+x\sqrt{2}}\right] + c \quad [\because t = 1/x]$$

Ex.6 Evaluate $I = \int \frac{dx}{2x\sqrt{1-x}\sqrt{(2-x)+\sqrt{1-x}}}$.

Sol.

$$\text{Here, } I = \int \frac{dx}{2x\sqrt{1-x}\sqrt{(2-x)+\sqrt{1-x}}}$$

$$I = \int \frac{2t\,dt}{2(1-t^2).t\sqrt{1+t^2+t}} = -\int \frac{dt}{(1-t^2)\sqrt{t^2+t+1}}$$

$$= \int \frac{dt}{(t-1)(t+1)\sqrt{t^2+t+1}} = \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) \frac{dt}{\sqrt{t^2+t+1}}$$

$$\because \quad \frac{1}{(t-1)(t+1)} = \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right)$$

$$= \frac{1}{2}\int \frac{1}{(t-1)\sqrt{t^2+t+1}}\ .dt - \frac{1}{2}\int \frac{1.dt}{(t+1)\sqrt{t^2+t+1}}$$

Let, $I = \frac{1}{2} I_1 - \frac{1}{2} I_2$

Where $I_1 = \int \frac{dt}{(t-1)\sqrt{t^2+t+1}}$ and

$I_2 = \int \frac{dt}{(t+1)\sqrt{t^2+t+1}}$ put $(t - 1) = \frac{1}{z}$ for I_1,

$$I_1 = \int \frac{-1/z^2 dz}{\frac{1}{z}\sqrt{\left(1+\frac{1}{z}\right)^2+\left(1+\frac{1}{z}\right)+1}} = -\int \frac{dz}{\sqrt{\left(z+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\log\left|\left(z+\frac{3}{2}\right)+\sqrt{z^2+3z+3}\right| \quad ....(ii)$$

For I_2, put $(t + 1) = \frac{1}{S}$, $I_2 = -\int \frac{dS}{\sqrt{\left(S-\frac{1}{2}\right)^2+\frac{3}{4}}}$

$$= -\log\left|\left(S-\frac{1}{2}\right)+\sqrt{S^2-S+1}\right| \quad ....(iii)$$

$$I = -\frac{1}{2}\left\{\log\left(z\frac{3}{2}+\sqrt{z^2+3z+3}\right)\right\} + \frac{1}{2}\log\left|\left(S-\frac{1}{2}\right)+\sqrt{S^2-S+1}\right| + c$$

where, $z = \frac{1}{\sqrt{1-x}-1}$ and $S = \frac{1}{\sqrt{1-x}+1}$.

Integration Of A Binomial Differential

The integral $\int x^m(a+bx^n)^p\,dx$, where m, n, p are rational numbers, is expressed through elementary functions only in the following three cases :

Case I : p is an integer. Then, if $p > 0$, the integrand is expanded by the formula of the binomial; but if $p < 0$, then we put $x = t^k$, where k is the common denominator of the fractions and n.

Case II : $\frac{m+1}{n}$ is an integer. We put $a + bx^n = t^\alpha$, where α is the denominator of the fraction p.

Case III : $\frac{m+1}{n} + p$ is an integer we put $a + bx^n = t^\alpha x^n$, where a is the denominator of the fraction p.

Ex.1 Evaluate $I = \int \sqrt[3]{x}(2+\sqrt{x})^2\,dx$.

Sol.

$$I = \int x^{\frac{1}{3}}(2+x^{\frac{1}{2}})^2\,dx.$$

Here $p = 2$, i.e. an integer, hence we have case I.

$$I = \int x^{\frac{1}{3}}(x+4x^{\frac{1}{2}}+4)\,dx = \int (x^{\frac{4}{3}}+4x^{\frac{5}{6}}+4x^{\frac{1}{3}})\,dx$$

$$= \frac{3}{7}x^{\frac{7}{3}} + \frac{24}{11}x^{\frac{11}{6}} + 3x^{\frac{4}{3}} + C.$$

Ex.2 Evaluate $I = \int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$.

Sol. $I = \int x^{-\frac{2}{3}}(1+x^{\frac{1}{3}})^{\frac{1}{2}}\,dx$.

Here $m = -\frac{2}{3}$; $n = \frac{1}{3}$; $p = \frac{1}{2}$;

$$\frac{m+1}{n} = \frac{\left(-\frac{2}{3}+1\right)}{\frac{1}{3}} = 1$$

i.e. an integer.we have case II. Let us make the substitution. Hence

, $I = 6\int t^2 dt = 2t^3 + C = 2(1+x^{\frac{1}{3}})^{\frac{3}{2}} + C$.

Ex.3 Evaluate I = $\int x^{-11}(1+x^4)^{-\frac{1}{2}}dx$.

Sol.

Here p = – 1/2 is a fraction, m+1/2 = -5/2 also a fraction, but m+1/n + p/2 = -5/2 -1/2 = -3 is an integer, i.e. we have case III, we put $1 + x^4 = x^{4/2}$,

Hence $x = \frac{1}{(t^2-1)^{1/4}}$; $dx = -\frac{tdt}{2(t^2-1)^{5/4}}$

Substituting these expression into the integral, we obtain

$$I = -\frac{1}{2}\int (t^2-1)^{\frac{11}{4}}\left(\frac{t^2}{t^2-1}\right)^{-\frac{1}{2}}\frac{tdt}{2(t^2-1)^{\frac{5}{4}}}$$

$$= -\frac{1}{2}\int (t^2-1)^2 dt = \frac{t^5}{10} + \frac{t^3}{3} - \frac{t}{2} + C.$$

Returning to x, we get I = $-\frac{1}{10x^{10}}\sqrt{(1+x^4)^5} + \frac{1}{3x^6}\sqrt{(1+x^4)^3} - \frac{1}{2x^3}\sqrt{1+x^4} + C$

Euler's Substitutions

Definition

Integrals of the form $\int R(x, \sqrt{ax^2+bx+c})dx$ are calculated with the aid of the three Euler substitutions.

1. $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}$ is $a > 0$;

2. $\sqrt{ax^2+bx+c} = t \pm \sqrt{c}$ if $a > 0$;

3. $\sqrt{ax^2+bx+c} = (x-\alpha)$ t if $ax^2 + bx + c = a(x-\alpha)(x-\beta)$

i.e. if α is real, I root of the trinomial $ax^2 + bx + c$.

Ex.1 Evaluate $I = \int \frac{xdx}{(\sqrt{7x-10-x^2})}$.

Sol. In this case $a < 0$ and $c < 0$ therefore neither the first, nor the second, Euler substitution is applicable.

But the quadratic trinomial $7x - 10 - x^2$ has real roots $\alpha = 2, \beta = 5$, therefore we use the third Euler substituion :

$\sqrt{7x-10-x^2} = \sqrt{(x-2)(5-x)} = (x-2)\,t.$

Whence $5 - x = (x-2)\,t^2$;

$x = \frac{5+2t^2}{1+t^3}$; $dx = -\frac{6tdt}{(1+t^2)^2}$;

$(x-2)\,t = \left(\frac{5+2t^2}{1+t^2} - 2\right) t = \frac{3t}{1+t^2}$.

Hence $I = -\frac{6}{27} \int \frac{5+2t^2}{t^2} dt = -\frac{2}{9} \int \left(\frac{5}{t^2} + 2\right) dt$

$= -\frac{2}{9}\left(\frac{-5}{t} + 2t\right) + C$. Where $t = \frac{\sqrt{7x-10-x^2}}{x-2}$.

Ex.2 Evaluate $\int (x + \sqrt{1+x^2})^n dx$.

Sol.

Let $I = \int (x + \sqrt{1+x^2})^n dx$ Put $x + \sqrt{1+x^2} = t$ ...(1)

$\left(1 + \frac{1}{2\sqrt{1+x^2}}.2x\right) dx = dt$ $\Rightarrow$ $\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right) dx = dt$ ...(2)

We know

$t = x + \sqrt{1+x^2} = x + \sqrt{1+x^2} \times \frac{x - \sqrt{1+x^2}}{x - \sqrt{1+x^2}}$

$t = \frac{-1}{x - \sqrt{1+x^2}}$ $\Rightarrow t = x + \sqrt{1+x^2}$

and $-\frac{1}{t} = x - \sqrt{1+x^2}$ substracting we get,

$2\sqrt{1+x^2} = t + \frac{1}{t}$ or $\frac{1}{\sqrt{1+x^2}} = \frac{2t}{t^2+1}$ ....(iii)

from (i), (ii) and (iii) we get $dx = \frac{t^2+1}{2t^2} dt$

$I = \int t^n . \frac{t^2+1}{(2t^2)} = \frac{1}{2}\int (t^n + t^{n-2}) dt = \frac{1}{2}\left[\frac{t^{n+1}}{n+1} + \frac{t^{n-1}}{n-1}\right] + c$

$\Rightarrow$ $I = \frac{1}{2(n+1)} [x + \sqrt{(1+x^2)}\,]^{n+1} + \frac{1}{2(n-1)} (x + \sqrt{(1+x^2)}\,)^{n-1} + c$

Can We Integrate All Continuous Function?

The question arises: Will our strategy for integration enables us to find the integral of every continuous function? For example, can we use it to evaluate $\int e^{x^2} dx$? The answer is no, at least not in terms of the functions that we are familiar with.

The functions that we have been dealing with in this book are called elementary functions. These are the polynomials, rational functions, power functions (x^3), exponential function (a^x), logarithmic functions trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction multiplication, division, and composition for instance, the function f(x) $= \sqrt{\frac{x^2-1}{x^3+2x-1}} + \ell n(\cosh x) - xe^{\sin 2x}$ is an elementary function

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by $F(x) = \int_0^x e^{t^2} dt$ then we know from part 1 of the fundamental theorem of calculus that $F'(x) = e^{x^2}$ Thus, $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function.

?This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2} dx$ in term of the function we know. The same can be said of the following integrals.

$\int \frac{e^x}{x} dx$ $\quad \int \sin(x^2) dx$ $\quad \int \cos(e^x) dx$

$\int \sqrt{x^3+1}\, dx$ $\quad \int \frac{1}{\ell n x} dx$ $\quad \int \frac{\sin x}{x} dx$

Derivative of Antiderivative (Leibniz Rule)

Rule:
If h(x) & g(x) are differentiable function of x

then, $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f[h(x)] \,.\, h'(x) - f[g(x)] \,.\, g'(x)$

Solved ExamplesEx.1 Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\, dt$.

Sol. Since $f(t) = \sqrt{1+t^2}$ is continuous, therefore $g'(x) = \sqrt{1+x^2}$

Ex.2 If $F(t) = \int_0^t \frac{1}{x^2+1} dx$, find F'(1), F'(2), and F'(x).

Sol. The integrand in this example is the continuous function f defined by f(x) $= \frac{1}{x^2+1}$.

$F'(t) = f(t) = \frac{1}{t^2+1}$. In particular,

$$F'(1) = \frac{1}{1^2+1} = \frac{1}{2}, \quad F'(2) = \frac{1}{2^2+1} = \frac{1}{5},$$

Ex.3 Find $\frac{d}{dx}\int_1^{x^4} \sec t\, dt$.

Sol. Let $u = x^4$. Then

$$\frac{d}{dx}\int_1^{x^4} \sec t\, dt = \frac{d}{dx}\int_1^{u} \sec t\, dt$$

$$= \frac{d}{du}\left(\int_1^{u} \sec t\, dt\right)\frac{du}{dx} = \sec u \frac{du}{dx} = \sec(x^4) \cdot 4x^3.$$

Ex.4 FInd the derivative of $F(x) = \int_{\pi/2}^{x^3} \cos t\, dt$

Sol.

$$F'(x) = \frac{dF}{du}\frac{du}{dx} = \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\, dt\right]\frac{du}{dx} = (\cos u)(3x^2) = (\cos x^3)(3x^2)$$

$$F(x) = \int_{\pi/2}^{x^3} \cos t\, dt = \sin t\Big]_{\pi/2}^{x^3}$$

$$= \sin x^3 - \sin\frac{\pi}{2} = (\sin x^3) - 1$$

$F'(x) = (\cos x^3)(3x^2)$.**Ex.5 Let** $f(x) = \int_0^x \{(a-1)(t^2+t+1)^2 - (a+1)(t^4+t^2+1)\}$. **Find the value of 'a' for which $f'(x) = 0$ has two distinct real roots.**

Sol. Differentiating the given equation, we get $f'(x) = (a-1)(x^2+x+1)^2 - (a+1)(x^2+x+1)(x^2-x+1)$.
Now, $f'(x) = 0 \Rightarrow (a-1)(x^2+x+1) - (a+1)(x^2-x+1) = 0 \Rightarrow x^2 - ax + 1 = 0$.
For distinct real roots $D > 0$ i.e. $a^2 - 4 > 0 \Rightarrow a^2 > 4 \Rightarrow a \in (-\infty, -2) \cup (2, \infty)$

Ex.6 Show that for a differentiable function f(x),

$$\int_0^n f'(x)\left\{[x]-x+\frac{1}{2}\right\}dx=\int_0^n f(x)dx+\frac{1}{2}f(0)+\frac{1}{2}f(n)-\sum_{r=0}^{n}f(r),$$

(where [*] denotes the greaetest integer function and n ε N)

Sol.

$$I=\int_0^n f'(x)[x]\,dx-\int_0^n xf'(x)\,dx+\frac{1}{2}\int_0^n f'(x)\,dx$$

$$=\sum_{r=1}^{n}\int_{r-1}^{r}f'(x)[x]\,dx-\left\{(xf(x))_0^n-\int_0^n f(x)dx\right\}+\frac{1}{2}(f(x))_0^n$$

$$=\sum_{r=1}^{n}(r-1)\int_{r-1}^{r}f'(x)\,dx-nf(n)+\frac{1}{2}f(n)-\frac{1}{2}f(0)+\int_0^n f(x)dx$$

$$=\sum_{r=1}^{n}(r-1)\{f(r)-f(r-1)\}-nf(n)+\frac{1}{2}f(n)+\int_0^n f(x)dx-\frac{1}{2}f(0)$$

$$=-f(1)-f(2)-\dots\dots-f(n-1)-f(n)+\frac{1}{2}f(n)+\frac{1}{2}f(0)+\int_0^n f(x)dx$$

$$=\sum_{r=1}^{n}f(r)+\frac{1}{2}f(n)+\frac{1}{2}f(0)+\int_0^n f(x)dx$$

Ex.7 Evaluate $\int_{-\infty}^{0} xe^x\,dx$.

Sol.

$$\int_{-\infty}^{0} xe^{x}\,dx = \lim_{t\to-\infty} \int_{t}^{0} xe^{x}\,dx$$

We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$;

$$\int_{t}^{0} xe^{x}\,dx = xe^{x}\Big|_{t}^{0} - \int_{t}^{0} e^{x}\,dx = -te^{t} - 1 + e^{t}$$

We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hopital's Rule we have

$$\lim_{t\to-\infty} te^{t} = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} (-e^{t}) = 0$$

Therefore $\int_{-\infty}^{0} xe^{x}\,dx = \lim_{t\to-\infty} (-te^{t} - 1 + e^{t}) = -0 - 1 + 0 = -1$

Ex.8 Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

Sol.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

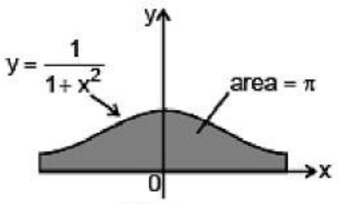
We must now evaluate the integrals on the right side separately :

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2} = \lim_{t\to\infty} \tan^{-1} x\Big]_{0}^{t}$$

$$= \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0) = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1} x\Big]_{t}^{0}$$

$$= \lim_{t\to\infty} (\tan^{-1} 0 - \tan^{-1} t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$



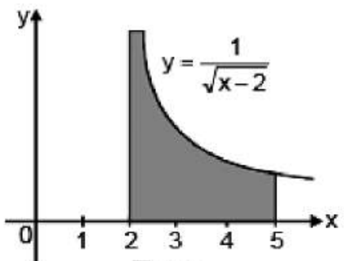
Figure

Since both of these integrals are convergent, the given integral is convergent and $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ Since $1/(1 + x^2) > 0$, the given improper integral can be interpreated as the area of the infinite region that lies under the curve $y = 1/(1 + x^2)$ and above the x-axis (see Figure).

Ex.9 Find $\int_2^5 \frac{1}{\sqrt{x-2}} dx$.

Sol.

We note first that the given integral is improper because $f(x) = 1/\sqrt{(x-2)}$ has the vertical asymptote $x = 2$. Since the infinite discontinuity occurs at the left end point of [2, 5]



Figure

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \left[2\sqrt{x-2}\right]_t^5$$

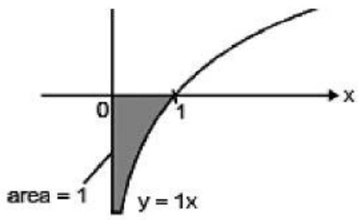
$$= \lim_{t\to 2^+} (2\sqrt{3} - \sqrt{x-2}) = 2\sqrt{3}$$

Thus, the given improper integrat is convergent and, since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure.

Ex.10 Evaluate $\int_0^1 \ln x \, dx$.

Sol. We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x\to 0^+} \ell n\, x = -\infty$. Thus, the given integral is improper and we have $\int_0^1 \ln x \, dx = \lim_{x\to 0^+} \int_t^1 \ln x \, dx$

Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$



Figure

$\int_t^1 \ln x\, dx = x \ln x]_t^1 - \int_t^1 dx$ $= 1 \ln - t \ln t - (1 - t) = - t \ln t - 1 + t$

To find the limit of the first term we use I'Hopital's Rule :

$\lim_{x\to 0^+} t \ln t = \lim_{x\to 0^+} \frac{\ln t}{1/t} = \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} (-t) = 0$

Therefore $\int_0^1 \ln x\, dx = \lim_{t\to 0^+} (-t \ln t - 1 + t)$ $= -0 - 1 + 0 = -1$

Figure shows the geometric interpretation of this result. The area of the shaded region above $y = \ln x$ and below the x-axis is 1.

Ex.11 Evaluate $\int_0^\infty [2e^{-x}]\, dx$, **(where [*] denotes the greatest integer function)**

Sol.

Let $I = \int_0^\infty [2e^{-x}]\, dx$. Let $y = 2e^{-x}$

$\frac{dy}{dx} = -2e^{-x} < 0 \ \forall\ x \in [0, \infty)$

$\therefore$ $2e^{-x}$ is decreaisng function $\forall\ x \in [0, \infty)$

$\Rightarrow 0 < 2e^{-x} \le 2 \ \forall\ x \in [0, \infty)$

for $x > \ln 2 \Rightarrow e^x > 2 \Rightarrow e^{-x} < 1/2 \Rightarrow 2e^{-x} < 1 \therefore 0 \le 2e^{-x} < 1$ $[2e^{-x}] = 0$

$\therefore\ I = \int_0^{\ln 2} [2e^{-x}]\, dx + \int_{\ln 2}^\infty [2e^{-x}]\, dx$

$= \int_0^{\ell n 2} 1.\,dx + \int_{\ell n 2}^{\infty} 0.dx = (\ell n\, 2 - 0) + 0 = \ell n\, 2$

Definite Integral As Limit Of A Sum And Estimate Of Definite Integrals

Definite Integral As Limit Of A Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots\ldots + f(a + \overline{n-1}h)]$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh) \quad \text{where } b - a = nh$$

If $a = 0$ **&** $b = 1$ **then,** $\lim_{n\to\infty} h \sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)\,dx$; **where** $nh = 1$

or $\lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

Remark :

The symbol $\int$ was introduced by Leibnitz and is called integral sign. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit. The symbol dx has no official meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The procedure of calculating an integral is called integration.

Estimate Of Definite Integration & General Inequality

STATEMENT : If f is continuous on the interval [a, b], there is atleast one number c between a and b such that $\int_a^b f(x)\,dx = f(c)\,(b - a)$

Proof : Suppose M and m are the largest and smallest values of f, respectively, on [a, b]. This means $m \le f(x) \le M$ when $a \le x \le b$

$$\Rightarrow \quad \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx \quad \text{Dominance rule}$$

$$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\Rightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

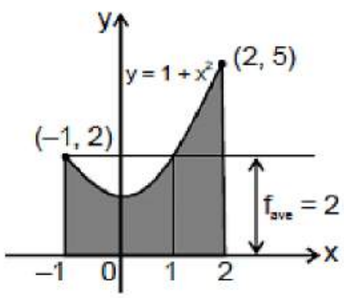
Because f is continuous on the closed interval [a, b] and because the number I $= \frac{1}{b-a}\int_a^b f(x)\,dx$ lies between m and M, the intermediate value theorem syas there exists a number c between a and b for which f(c) = I ; that is, $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$ $\quad \int_a^b f(x)\,dx = f(c)\,(b-a)$

The mean value theorem for integrals does not specify how to determine c. It simply guarantees the existence of atleast one number c in the interval.
Since $f(x) = 1 + x^2$ is continuous on the interval [–1, 2], the Mean Value Theorem for Integrals says there is a number c in [–1, 2] such that $\int_{-1}^{2}(1+x^2)\,dx = f(c)\,[2-(-1)]$

In this particular case we can find c explicitly. From previous Example we know that fave = 2, so the value of c satisfies $f(c) = f_{ave} = 2$

Therefore $\quad 1 + c^2 = 2 \quad$ so $\quad c^2 = 1$

Thus, in this case there happen to be two numbers $c = \pm 1$ in the interval [–1, 2] that work in the mean value theorem for Integrals.



Figure

Walli's Formula & Reduction Formula

$$\int_0^{\pi/2} \sin^n x . \cos^m x \, dx = \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 3][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)... 1 \text{ or } 2} K$$

Where K $= \frac{\pi}{2}$ **if both m and n are even** $(m, n \in N)$ **;**

= 1 otherwise

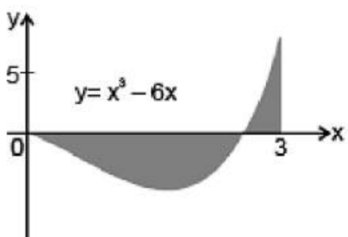
Solved Examples

Ex.1 Evaluate $\int_0^3 (x^3 - 6x) dx$ using limit of sum.

Sol.

$$\int_0^3 (x^3 - 6x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$



Figure

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right] = \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2} \frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$

This integral can't be interpreated as an area because f takes on both positive and negative values. But is can be interpreated as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in Figure

Ex.2 Prove that, $\int \sin n\theta \sec\theta \, d\theta = -\frac{2\cos(n-1)\theta}{n-1} - \int \sin(n-2)\theta \sec\theta \, d\theta$. **Hence or otherwise evaluate** $\int_0^{\pi/2} \frac{\cos 5\theta \sin 3\theta}{\cos\theta} d\theta$.

Sol.

Consider $\sin n\theta + \sin(n-2)\theta = 2\sin(n-1)\theta\cos\theta \Rightarrow \sin n\theta \sec\theta = 2\sin(n-1)\theta - \sin(n-2)\theta\sec\theta$

Hence $\int \sin n \sec\theta \, d\theta$

$$= -\frac{2}{(n-1)}\cos(n-1)\theta - \int \sin(n-2)\sec\theta \, d\theta$$

Now $\frac{1}{2}\int_0^{\pi/2} \frac{2\sin 3\theta \cos 5\theta}{\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{\sin 8\theta - \sin 2\theta}{\cos\theta}$, $I = \frac{1}{2} I_8 - 1$

$$I_8 = -\frac{2}{7}\cos 7\theta\Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 6\theta}{\cos\theta} d\theta.$$

$$= \frac{2}{7} - \left[-\frac{2}{5}\cos 5\theta\Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 4\theta}{\cos\theta} d\theta\right]$$

$$= \frac{2}{7} - \left[\frac{2}{5} - \left\{-\frac{2}{3}\cos 3\theta\Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 2\theta}{\cos\theta} d\theta\right\}\right]$$

$$= \frac{2}{7} - \left[\frac{2}{5} - \frac{2}{3} + 2\right] = \frac{2}{7} - \frac{2}{5} + \frac{2}{3} - \frac{2}{1}$$

$$= \frac{30 - 42 + 70 - 210}{105} = -\frac{152}{105}$$

$I = -\frac{152}{2\times105} - 1 = -\frac{76+105}{105} = -\frac{181}{105}$

Ex.3 Prove that $\int_0^{\pi/4} (\cos 2\theta)^{3/2}\cos\theta\, d\theta = \frac{3\pi}{16\sqrt{2}}$

Sol.

L.H.S. = $\int_0^{\pi/4} (\cos 2\theta)^{3/2}\cos\theta\, d\theta = \int_0^{\pi/4} (1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta$

(Put $\sqrt{2}\sin\theta = \sin t \Rightarrow \cos\theta\, d\theta = \frac{\cos t}{\sqrt{2}}\, dt$)

when $\theta \to 0$ then $t \to 0$; $\theta \to \pi/4$ then $t \to \pi/2$

$\therefore$ L.H.S. = $\int_0^{\pi/2} \frac{\cos^4 t}{\sqrt{2}}\, dt = \frac{1}{\sqrt{2}}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{16\sqrt{2}}$ = R.H.S. (From Walli's formula)

Ex.4 If $u_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\, dx$, **then show that** $u_1, u_2, u_3, \ldots$ **constitute an arithmetic progression. Hence or otherwise find the value of** u_n.

Sol.

$u_{n+1} - 2u_n + u_{n-1} = (u_{n+1} - u_n) - (u_n - u_{n-1})$

$= \int_0^{\pi/2} \frac{(\sin^2(n+1)x - \sin^2 nx) - (\sin^2 nx - \sin^2(n-1)x}{\sin^2 x}\, dx$

$= \int_0^{\pi/2} \frac{(\sin(2n+1)x\sin x - \sin(2n-1)x\sin x)}{\sin^2 x}\, dx$

$= \int_0^{\pi/2} \frac{(\sin(2n+1)x - \sin(2n-1)x)}{\sin x} = \int_0^{\pi/2} \frac{2\cos 2nx \sin x}{\sin x}$

$= 2\int_0^{\pi/2} \cos 2nx\, dx = 2\,.\,\frac{\sin 2nx}{2n}\Big|_0^{\pi/2}$

$= \frac{1}{n}(\sin n\pi - \sin 0) = 0 - 0 = 0$

$\therefore u_{n-1} + u_{n+1} = 2u_n$ i.e., u_{n-1}, u_n, u_{n+1} form an A.P.

$\Rightarrow u_1, u_2, u_3,$.........constitute an A.P.

Ex.5 Evaluate $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx$.

Sol.

Let $I = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$

$= \int_0^1 \cot^{-1}(1 - x(1 - x))dx$

$= \int_0^1 \tan^{-1}\left(\frac{1}{1-x(1-x)}\right)dx = \int_0^1 \tan^{-1}\left(\frac{x+(1+x)}{1-x(1-x)}\right)dx$

$= \int_0^1 (\tan^{-1} x + \tan^{-1}(1-x))\, dx$

$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$

$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-(1-x))\, dx = 2\int_0^1 \tan^{-1} x\, dx$

Integrating by parts taking unity as the second function, we have

$I = 2\left[\left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx\right] = 2\left[\frac{\pi}{4} - \frac{1}{2}[\ln|1+x^2|]_0^1\right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}\ln 2\right]$ Hence $I = \frac{\pi}{2} - \ln 2$.

Ex.6 Show that $\int_0^\pi \frac{dx}{(a - \cos x)} = \frac{\pi}{\sqrt{(a^2-1)}}$. **Hence or otherwise evaluate** $\int_0^\pi \frac{dx}{(\sqrt{5} - \cos x)^3}$.

Sol.

Let $I = \int_0^\pi \frac{dx}{(a - \cos x)}$ ....(1)

$= \int_0^\pi \frac{dx}{a - \cos(\pi - x)}$ (By Prop.)

$= \int_0^{\pi} \frac{dx}{(a+\cos x)}$ ....(2)

adding (1) and (2) then

$$2I = \int_0^{\pi} \frac{2a\,dx}{(a^2-\cos^2 x)} = 2a\,.\,2\int_0^{\pi/2} \frac{dx}{(a^2-\cos^2 x)}$$

$$\Rightarrow \quad I = 2a\int_0^{\pi/2} \frac{dx}{(a^2-\cos^2 x)}$$

$$= 2a\int_0^{\pi/2} \frac{\sec^2 dx}{a^2(1+\tan^2 x)-1}$$

$$= 2a\int_0^{\pi/2} \frac{\sec^2 x\,dx}{(a^2-1)+(a\tan x)^2}$$

Put a tan x = t ⇒ a sec²x dx = dt when x = 0 ⇒ t = 0 ; x = p/2 ⇒ t = ∝

$$\text{then } I=2\int_0^{\infty} \frac{dt}{(\sqrt{a^2-1})^2+t^2} = \frac{2}{\sqrt{(a^2-1)}}\left\{\tan^{-1}\left(\frac{t}{\sqrt{(a^2-1)}}\right)\right\}_0^{\infty}$$

$$= \frac{2}{\sqrt{(a^2-1)}}\{\tan^{-1}\infty - \tan^{-1}0\} = \frac{2}{\sqrt{(a^2-1)}}\left\{\frac{\pi}{2}-0\right\}$$

Hence $I = \frac{\pi}{\sqrt{(a^2-1)}}$ or $\int_0^{\pi} \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{(a^2-1)}}$

Differentiating both side w.r.t. 'a', we get $-\int_0^{\pi} \frac{dx}{(a-\cos x)^2} = \frac{-\pi a}{(a^2-1)^{3/2}}$

again differentiating both sides w.r.t. 'a' we get $2\int_0^{\pi} \frac{dx}{(a-\cos x)^3} = \frac{\pi(2a^2+1)}{(a^2-1)^{3/2}}$

Put a = √5 on both sides, we get

$$2\int_0^{\pi} \frac{dx}{(\sqrt{5}-\cos x)^3} = \frac{\pi(11)}{(4)^{3/2}} \quad \text{or} \quad \int_0^{\pi} \frac{dx}{(\sqrt{5}-\cos x)^{3/2}} = \frac{11\pi}{16}$$

Ex.7 Let f be an injective functions such that f(x) f(y) + 2 = f(x) + f(y) + f(xy) for all

non negative real x and y with $f(0) = 1$ and $f'(1) = 2$ find f(x) and show that $\int f(x) dx - x(f(x) + 2)$ is a constant.

Sol. We have $f(x) f(y) + 2 = f(x) + f(y) + f(xy)$ ......(1)

Putting $x = 1$ and $y = 1$ then $f(1) f(1) + 2 = 3f(1)$

we get $f(1) = 1, 2$ & $f(1) \neq 1$ ($\because f(0) = 1$ & function is injective) then $f(1) = 2$

Replacing y by 1/x in (1) then $f(x) f(1/x) + 2 = f(x) + f(1/x) + f(1) \Rightarrow f(x) f(1/x) = f(x) + f(1/x)$ [$f(1) = 2$)

Hence f(x) is of the type $f(x) = 1 \pm x^n \Rightarrow f(1) = 1 \pm 1 = 2$ (given)

$\therefore f(x) = 1 + x^n$ and $f'(x) = nx^{n-1} \Rightarrow f'(1) = n = 2 \therefore f(x) = 1 + x^2$

$\therefore \; 3\int f(x)\,dx - x(f(x) + 2) = 3\int (1 + x^2)dx - x(1 + x^2 + 2)$

$= 3\left(x + \frac{x^3}{3}\right) - x(3 + x^2) + c$

Ex.8 Evaluate $\int_0^{\pi} ||\sin x| - |\cos x||\,dx$

Sol.

Let $I = \int_0^{\pi} ||\sin x| - |\cos x||\,dx$ Make $|\sin x| - |\cos x| = 0 \therefore |\tan x| = 1$

$\therefore \tan x = \pm 1 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$ and both these values lie in the interval $[0, \pi]$.

We find for $0 < x < \pi/4$, $|\sin x| - |\cos x| < 0$

$\frac{\pi}{4} < x < \frac{3\pi}{4}$, $|\sin x| - |\cos x| > 0$

$= -\int_0^{\pi/4} \sin x\,dx + \int_0^{\pi/4} \cos x\,dx + \int_{\pi/4}^{3\pi/4} \sin x\,dx - \int_{\pi/4}^{\pi/2} \cos x\,dx$

$+ \int_{\pi/2}^{3\pi/2} \cos x\,dx - \int_{\pi/4}^{\pi} \sin x\,dx - \int_{3\pi/4}^{\pi} \cos x\,dx$

$$= \left(\frac{1}{\sqrt{2}} - 1\right) + \left(\frac{1}{\sqrt{2}} - 0\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - 1\right) + \left(-1 + \frac{1}{\sqrt{2}}\right) - \left(0 - \frac{1}{\sqrt{2}}\right) = 4\sqrt{2} - 4$$

Ex.9 Evaluate $\int_0^2 [x^2 - x + 1]\, dx$**, (where [*] is the greatest integer function)**

Sol. Let I = $\int_0^2 [x^2 - x + 1]\, dx$

Let f(x) = $x^2 + x + 1 \Rightarrow f'(x) = 2x - 1$ for $x > 1/2$, $f'(x) > 0$ and $x < 12$, $f'(x)$, 0

Values of f(x) at x = 1/2 and 2 are 3/4 and 3 integers between them an 1, 2 then $x^2 - x + 1 = 1, 2$

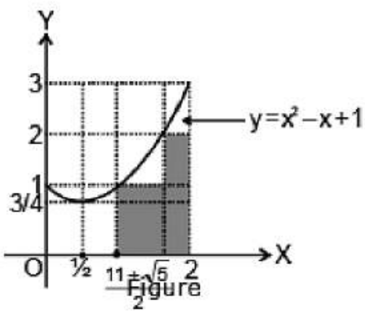
we get x = 1, $x = \frac{1+\sqrt{5}}{2}$ and values of f(x) at x = 0 and 1/2 are 1 and 3/4 no integer between them

$$I = \int_0^{1/2} [x^2 - x + 1]\, dx + \int_{1/2}^{1} [x^2 - x + 1]\, dx + \int_1^{\frac{1+\sqrt{5}}{2}} [x^2 - x + 1]\, dx + \int_{\frac{1+\sqrt{5}}{2}}^{2} [x^2 - x + 1]\, dx$$

$$= 0 + 0 + 1\int_1^{\frac{1+\sqrt{5}}{2}} 1.dx + 2\int_{\frac{1+\sqrt{5}}{2}}^{2} 1\, dx$$

$$= \left(\frac{1+\sqrt{5}}{2} - 1\right) + 2\left(2 - \frac{1+\sqrt{5}}{2}\right) = \left(\frac{5-\sqrt{5}}{2}\right)$$

Alternative Method : It is clear from the figure



Figure

$\int_0^2 [x^2 - x + 1]\, dx$ = Area of bounded region

$= 0 + \left(\frac{1+\sqrt{5}}{2} - 1\right) \times 1 + \left(2 - \frac{1+\sqrt{5}}{2}\right) \times 2$

$= 3 - \left(\frac{1+\sqrt{5}}{2}\right) = \left(\frac{1-\sqrt{5}}{2}\right)$

Ex.10 If $\int_0^\pi \left(\frac{x}{1+\sin x}\right)^2 dx = \lambda$ then show that $\int_0^x \frac{2x^2\cos^2(x/2)}{(1+\sin x)^2} dx = \lambda + 2\pi - \pi^2$.

Sol.

Let $\int_0^\pi \frac{2x^2\cos^2(x/2)}{(1+\sin x)^2} dx = \int_0^\pi \frac{x^2(1+\cos x)}{(1+\sin x)^2} dx$

$= \lambda + \int_0^\pi x^2 \frac{\cos x}{(1+\sin x)^2} dx$

Integrating by parts taking x^2 as 1 st function, we we get $= \lambda + \left[x^2\left\{\frac{1}{(1+\sin x)}\right\}\right]_0^\pi + 2\int_0^\pi \left(\frac{x}{1+\sin x}\right) dx$ ...(1)

$I = \lambda - \pi^2 + 2\int_0^\pi \frac{x}{1+\sin x} dx$ [By Prop.]

$= \lambda - \pi^2 + 2\int_0^\pi \frac{(\pi - x)dx}{1+\sin x}$ ....(2)

Adding (1) and (2) we get

$2I = 2\lambda - 2\pi^2 + 2\pi \int_0^\pi \frac{x}{(1+\sin x)}$

or $I = \lambda - \pi^2 + \pi \int_0^\pi \frac{(1-\sin x)}{1-\sin^2 x} dx$

$= \lambda - \pi^2 + \pi \int_0^\pi (\sec^2 x - \sec x \tan x) dx$

$= \lambda - \pi^2 + \pi\, \{\tan x - \sec x\}_0^{\pi}$

$= \lambda - \pi^2 + \pi\, \{(0+1)-(0-1)\}$

$= \lambda - \pi^2 + 2\pi$ Hence $I = \lambda - 2\pi + \pi^2$

Ex.11 Prove that $\int_0^{\pi/2} \frac{x \sin x \cos x}{\left(a^2 \cos^2 x + b^2 \sin^2 x\right)} dx = \frac{\pi}{4ab^2(a+b)}$

Sol.

Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\left(a^2 \cos^2 x + b^2 \sin^2 x\right)} dx$ Intergrating by parts taking x as a first function, we have

$$= \left[x\left\{\frac{-1}{2\left(b^2-a^2\right)\left(a^2\cos^2 x + b^2 \sin^2 x\right)}\right\}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{2\left(b^2-a^2\right)\left(a^2\cos^2 x + b^2\sin^2 x\right)} dx$$

$$= -\frac{\pi}{4\left(b^2-a^2\right)b^2} + \frac{1}{2\left(b^2-a^2\right)} \int_0^{\pi} \frac{\sec^2 x dx}{a^2 + (b \tan x)^2}$$

$$= \frac{-\pi}{4\left(b^2-a^2\right)b^2} + \frac{1}{2\left(b^2-a^2\right)} \int_0^{\infty} \frac{dt}{b\left(a^2+t^2\right)}$$

$$= \frac{-\pi}{4\left(b^2-a^2\right)b^2} + \frac{1}{2ab\left(b^2-a^2\right)}\left[\tan^{-1}\left(\frac{t}{a}\right)\right]_0^{\infty}$$

$$= \frac{-\pi}{4\left(b^2-a^2\right)b^2} + \frac{1}{2ab\left(b^2-a^2\right)}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{-\pi}{4\left(b^2-a^2\right)b^2} + \frac{1}{4ab\left(b^2-a^2\right)} = \frac{\pi(b-a)}{4ab^2\left(b^2-a^2\right)}$$

Ex.12 Evaluate $\int_{-3/2}^{2} f(x)\, dx$, where f(x) is given by $f(x) = \max_{-3/2 \le t \le x} (|t-1| - |t| + t + 1)$

Sol.

Let $g(t) = |t-1| - |t| + |t+1| = \begin{cases} -t, & t<-1 \\ t+2, & -1<t<0 \\ 2-t, & 0<t<t \\ t, & t>1 \end{cases}$ Hence $\begin{cases} 3/2, & -3/2 \le x < -1/2 \\ 2+x, & -1/2 < x \le 0 \\ 2, & 0 < x \le 2 \end{cases}$

$\therefore \int_{-3/2}^{2} f(x) = \int_{-3/2}^{-1/2} 3/2\, dx + \int_{-1/2}^{0} (2+x)\, dx + \int_{0}^{2} 2dx$

$= \frac{3}{2}\left(-\frac{1}{2}+\frac{3}{2}\right)+0\left(-1+\frac{1}{8}\right) + 2\,(2-0) = \frac{51}{8}$

Ex.13 For all positive integer k, prove that $\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + \ldots\ldots + \cos(2k-1)x]$

Hence prove that $\int_0^{\pi/2} \sin 2kx \cot x\, dx = \pi/2$

Sol. We have 2 sinx [cos x + cos 3x + ...+ cos (2k–1)x]
= 2 sinx cosx + 2 sinx cos 3x + ...+ 2 sinx cos(2k – 1)x
= sin 2x + sin 4x – sin 2x + sin 6x – sin 4x + ... + sin 2kx – sin (2k – 2) x
= sin 2kx
⇒ 2[cos x + cos 3x + ... + cos (2k – 1) x] = sin2kx/sinx

$= 2\int_0^{\pi/2} \sin 2kx \cot x\, dx = \int_0^{\pi/2} \left(\frac{\sin 2kx}{\sin x}\right)\cos x\, dx$

$= \int_0^{\pi/2} [2\cos^2 x + 2\cos 3x \cos x + \ldots\ldots + 2\cos(2k-1)x \cos x]\, dx$

$= \int_0^{\pi/2} [\cos x + \cos 3x + \ldots\ldots + \cos(2k-1)x]\cos x\, dx$

$= \left[x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + \ldots\ldots + \frac{\sin 2kx}{2k} + \frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2} = \frac{\pi}{2}$

Ex.14 Let f(x) is periodic function such that $\int_0^x (f(t))^3 dt = \frac{1}{x^2}\left(\int_0^x (f(t)dt)\right)^3 \ \forall x \in R - \{0\}$ **Find the function f(x) if (1) = 1 .**

Sol.

Let $\int_0^x (f(t))^3 dt = F(x)$

$\Rightarrow f(x) = F'(x)$ ....(1)

$\therefore \int_0^x (f(t))^3 dt = \int_0^x \{F'(t)\}^3 dt$ ....(2)

and $\frac{1}{x^2}\left(\int_0^x (f(t)dt)\right)^3 = \frac{(F(x))^3}{x^2}$ ....(3)

from (2) and (3) $\int_0^x (F'(t))^3 dt = \frac{1}{x^2}\int_0^x (F(x))^3$

Differentiating bot sides w.r.t.x, we get

$F'(x))^3 = \frac{x^2.3(F(x))^2F'(x) - (F(x))^3.2x}{x^4} = \frac{3(F(x))^2F'(x) - (F(x))^3}{x^3}$

or $(x F'(x))^3 = 2x(F(x))^2F'(s) - 2(F(x))^3$

or $\left\{\frac{xF'(x)}{F(x)}\right\}^3 = 3\left\{\frac{xF'(x)}{F(x)}\right\} - 2$

$\Rightarrow \lambda^3 - 3\lambda + 2 = 0$

$\Rightarrow (\lambda-1)^2(\lambda+2) = 0$

for $\lambda = 1$

where $\lambda = \frac{xF'(x)}{F(x)}$

$\therefore \lambda = 1, -2,$

$\frac{xF'(x)}{F(x)} = 1$

$\Rightarrow \frac{F'(x)}{F(x)} = \frac{1}{x}$

$\Rightarrow F(x) = cx$

$f(x) = c$

$f(1) = 1 = c$

$f(x) = 1$

$\therefore \quad \ln F(x) = \ln x + \ln c$

$\therefore \quad \ln F'(x) = c$

for $\lambda = -2$: $\quad \dfrac{xF'(x)}{F(x)} = -2$

$\Rightarrow \quad \dfrac{xF'(x)}{F(x)} - \dfrac{2}{x}$

$\Rightarrow \quad F(x) = \dfrac{c_1}{x^2}$

$\therefore \quad \ln F(x) = -2\ln x + \ln c_1$

$\therefore \quad \ln F'(x) = -\dfrac{2c_1}{x^3}$

$\Rightarrow \quad f(x) = -\dfrac{2c_1}{x^3}$

$\Rightarrow \quad f(1) = 1 = -2c_1$

then $f(x) = 1/x^3$ But given $f(x)$ is a periodic function Hence $f(x) = 1$

Ex.15 Assume $\int_0^{\pi} \ln \sin\theta d\theta = -\pi \ln 2$ **then prove that**

$\int_0^{\pi} \theta^3 \ln \sin\theta \, d\theta = \dfrac{3\pi}{2} \int_0^{\pi} \theta^2 \ln(\sqrt{2}\sin\theta)\, d\theta.$

Sol.

Let $I = \int_0^{\pi} \theta^3 \ln \sin\theta \, d\theta$ ....(1)

$= \int_0^{\pi} (\pi-\theta)^3 \ln \sin\theta \, d\theta.$ [By Prop.]

$= \int_0^{\pi} (\pi^3 - 3\pi^2\theta + 3\pi\theta^2 - \theta^3) \ln \sin\theta \, d\theta$

$= \pi^3 \int_0^{\pi} \theta^3 \ln \sin\theta \, d\theta - 3\pi^2 \int_0^{\pi} \theta \ln \sin\theta \, d\theta + 3\pi \int_0^{\pi} \theta^2 \ln \sin\theta \, d\theta - \int_0^{\pi} \theta^3 \ln \sin\theta \, d\theta$

$= \pi^3 \int_0^{\pi} \theta^3 \ln \sin\theta \, d\theta - 3\pi^2 \int_0^{\pi} \theta \ln \sin\theta \, d\theta + 3\pi \int_0^{\pi} \theta^2 \ln \sin\theta \, d\theta - I$ [From (1)]

$\therefore 2I = \pi^3 I_1 - 3\pi^1 I_2 + 3\pi \int_0^{\pi} \theta^2 \ln \sin\theta \, d\theta$

$I_2 = \int_0^{\pi} \theta \ln \sin\theta \, d\theta = \int_0^{\pi} (\pi - \theta) \ln \sin(\pi - \theta) \, d\theta = \int_0^{\pi} (\pi - \theta) \ln \sin\theta \, d\theta$

$\therefore \quad 2I_2 = \pi \int_0^{\pi} \ln \sin\theta \, d\theta = -\pi^2 \ln 2$

$\therefore \quad I_2 = -\frac{\pi^2}{2} \ln 2$

then $2I = -\pi^2 \ln 2 \; \frac{3\pi^4}{2} \ln 2 + 3\pi \int_0^{\pi} \theta^2 \ln \sin\theta \, d\theta$

$\Rightarrow I = \frac{\pi^4}{2} \ln 2 \; \frac{3\pi^4}{2} \int_0^{\pi} \theta^2 \ln \sin\theta \, d\theta$

$= \frac{3\pi}{2} \int_0^{\pi} \theta^2 \ln \sqrt{2} \, d\theta = \frac{3\pi}{2} \int_0^{\pi} \theta^2 \ln \sin q \, d\theta$

$= \frac{3\pi}{2} \int_0^{\pi} \theta^2 \ln (\sqrt{2} \sin\theta) d\theta$

Ex.16 Use induction to prove that, $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \; \forall n \geq 2, \; n \in N$

Sol.

$P(k) : \int_0^{\pi/2} \cos^{k-2} x \sin kx \, dx = \frac{1}{k-1}$

$P(k+1) ; \int_0^{\pi/2} \cos^{k-1} x \sin(k+1) dx = \frac{1}{k}$

$= \int_0^{\pi/2} \cos^{k-2} x \sin(k+1) x \cos x \, dx$

Now $\sin kx = \sin[(k+1)x - x] = \sin(k+1)x \cos - \cos(k+1)x \sin x$

Hence $\sin(k+1)x \cos x = \sin kx + \cos(k+1)x \sin x$

Subisting P(k + 1) = $\int_0^{\pi/2} \cos^{k-2} [\sin kx + \cos(k+1) x \sin x]\, dx$

$P(k) + \int_0^{\pi/2} \cos^{k-2} \sin x \,.\, \cos(k+1) x\, dx$

Now I. B. P. to get the result

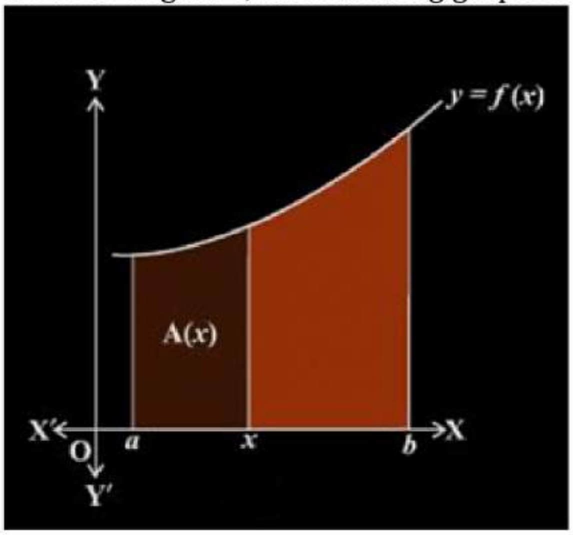
Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

The fundamental theorem of calculus is a theorem that links the concept of integrating a function with that of differentiating a function. The fundamental theorem of calculus justifies the procedure by computing the difference between the antiderivative at the upper and lower limits of the integration process. In this article, let us discuss the first, and the second fundamental theorem of calculus, and evaluating the definite integral using the theorems in detail.

Area Function

Let's consider a function f in x that is defined in the interval [a, b]. The integral of f(x) between the points a and b i.e. $\int_a^b f(x)d(x),$ is the area that is bounded by the curve y = f(x) and the lines x = a, x =b and x –axis $\int_a^x f(x)dx.$ depicts the area of the region shaded in brown where x is a point lying in the interval [a, b]. Assuming that the values taken by this function are non- negative, the following graph depicts f in x.



A(x) is known as the area function which is given as;

$$A(x) = \int_a^x f(x)\,dx$$

Depending upon this, the fundamental theorem of Calculus can be defined as two theorems as stated below:

First Fundamental Theorem of Integral Calculus (Part 1)

The first part of the calculus theorem is sometimes called the first fundamental theorem of calculus. It affirms that one of the antiderivatives (may also be called indefinite integral) say F, of some function f, may be obtained as integral of f with a variable bound of integration. From this, we can say that there can be antiderivatives for a continuous function.

Statement: Let f be a continuous function on the closed interval [a, b] and let A(x) be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.
Or

Let f be a continuous real-valued function defined on a closed interval [a, b]. Let F be the function defined, for all x in [a, b], by:

$$F(x) = \int_a^x f(t)dt$$

Then F is uniformly continuous on [a, b] and differentiable on the open interval (a, b), and $F'(x) = f(x)\ \forall\ x \in (a, b)$
Here, the $F'(x)$ is a derivative function of $F(x)$.

Second Fundamental Theorem of Integral Calculus (Part 2)

The second fundamental theorem of calculus states that, if the function "f" is continuous on the closed interval [a, b], and F is an indefinite integral of a function "f" on [a, b], then the second fundamental theorem of calculus is defined as:
$F(b) - F(a) = \int_a^b f(x)\, dx$
Here R.H.S. of the equation indicates the integral of f(x) with respect to x.
f(x) is the integrand.
dx is the integrating agent.
'a' indicates the upper limit of the integral and 'b' indicates a lower limit of the integral.
The function of a definite integral has a unique value. The definite integral of a function can be described as a limit of a sum. If there is an antiderivative F of the function in the interval [a, b], then the definite integral of the function is the difference between the values of F, i.e., F(b) – F(a).

Remarks on the Second Fundamental Theorem of Calculus

- The second part of the fundamental theorem of calculus tells us that $\int_a^b f(x)\, dx$ = (value of the antiderivative F of "f" at the upper limit b) – (the same antiderivative value at the lower limit a).
- This theorem is very beneficial because it provides us with a method of estimating the definite integral more quickly, without determining the sum's limit.
- In estimating a definite integral, the essential operation is finding a function whose derivative is equal to the integrand. However, this process will reinforce the relationship between differentiation and integration.
- In the expression $\int_a^b f(x)\, dx$, the function f(x) or say "f" should be well defined and continuous in the interval [a, b].

How to Calculate Definite Integral?

Here are the steps for calculating $\int_a^b f(x)dx$

- Determine the indefinite integral of f(x) as F(x). It must be noticed that arbitrary constant is not considered while calculating definite integrals since it cancels out itself, i.e.,

$$\int_a^b f(x)dx = [F(x)+C]_a^b = [F(b)+C] - [F(a)+C] = F(b) - F(a)$$

- Calculate F(b) – F(a) which gives us the value of the definite integral of f in x lying between the closed interval [a, b].

Examples

Q.1: Evaluate the integral: $\int_2^3 y^2dy$
Solution: Let $I = \int_2^3 y^2dy$
As we know,
$\int y^2dy = y^3/3 = F(y)$
Therefore, by second fundamental calculus theorem, we know;
I = F(3) – F(2) = 27/3 – 8/3 = 19/3

Q.2: Evaluate the integral: $\int_1^2[ydy/(y+1)(y+2)]$
Solution: By partial fraction we can factorise the term under integral.
y/[(y+1)(y+2)] = [-1/(y+1)]+[2/(y+2)] So,
∫y/[(y+1)(y+2)] = -log|y+1|+2log|x+2| = F(y)
Hence, by fundamental theorem of calculus part 2, we get;
I = F(2)-F(1) = [– log 3 + 2 log 4] – [– log 2 + 2 log 3] I = – 3 log 3 + log 2 + 2 log 4
I = log(32/27)

Practice Problems

Get more questions here for practice to understand the concept quickly.
Evaluate using the fundamental theorem of calculus:

1. $\int_0^{\frac{\pi}{4}} sin^3 2t\ cos2t\ dt$
2. $\int_3^4 \frac{1}{x} dx$
3. $\int_0^2 \frac{6x+3}{x^2-4} dx$

Frequently Asked Questions

What is the first fundamental theorem of calculus?

First fundamental theorem of integral calculus states that "Let f be a continuous function on the closed interval [a, b] and let A (x) be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$".

How many fundamental theorems of calculus are there?

Two basic fundamental theorems have been given in calculus for calculating the area using definite integrals:

First fundamental theorem of integral calculus

Second fundamental theorem of integral calculus

What are the 4 concepts of calculus?

The 4 concepts of calculus are:

Limits and functions

Derivatives

Integrals

Infinite series

What is the second fundamental theorem of calculus?

Second fundamental theorem of integral calculus states that "Let f be a continuous function defined on the closed interval [a, b] and F be an antiderivative of f. Then $\int_{a}^{b} f(x)\, dx = [F(x)]_{a}^{b} = F(b) - F(a)$.

Who first proved the fundamental theorem of calculus?

The first published statement and proof of a basic form of the fundamental theorem, strongly geometric, was given by James Gregory. Isaac Barrow proved a more generalized version of the theorem, while his student Isaac Newton finished the development of the enclosing mathematical theory.